$\alpha+$

Fantastic work!

String Theory I

Problem Set 1

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Th 11.00 – 12.30 C3 Wks 2, 4, 6, 8

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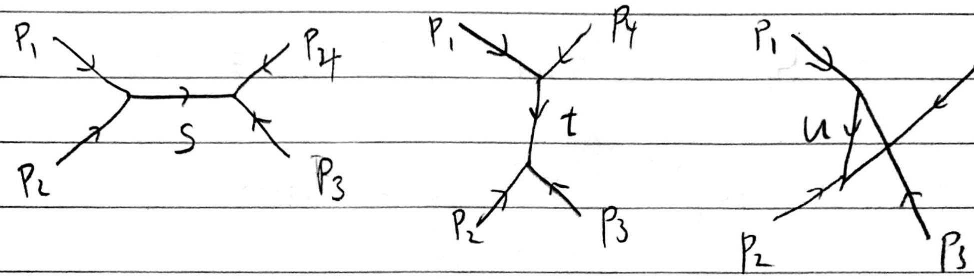
Q1

1) Mandelstam variables

$s = -(P_1+P_2)^2 \ , \ t = -(P_1+P_4)^2 \ , \ u = -(P_1+P_3)^2$



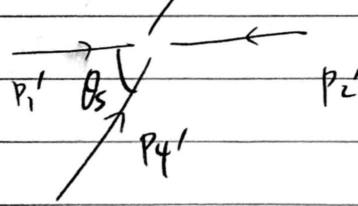
particles have identical mass $\alpha' m^2 = -1$

~~In~~ incoming momenta $P_1 = \begin{pmatrix} E_1 \\ \vec{P_1} \end{pmatrix}$ $P_2 = \begin{pmatrix} E_2 \\ \vec{P_2} \end{pmatrix}$.

outgoing momenta $P_3 = \begin{pmatrix} -E_3 \\ -\vec{P_3} \end{pmatrix}$ $P_4 = \begin{pmatrix} -E_4 \\ -\vec{P_4} \end{pmatrix}$.

with $E_j = \sqrt{|\vec{P_j}|^2 + m^2}$ $j = 1, \cdots, 4$.

In the CM frame S'



P_j' is the 4-momentum of P_j in CM frame.

Consider $P_1' \cdot P_4' = -(E_1')(-E_4') + \vec{P_1'} \cdot (-\vec{P_4'})$

$= E_1' E_4' - \vec{P_1'} \cdot \vec{P_4'}$

$= E_1' E_4' - |\vec{P_1'}||\vec{P_4'}| \cos\theta_s$

$\because$ CM frame and identical mass ✓

$\therefore$ ~~$|\vec{p}_j'| = |\vec{p}'|$ for p' f~~ $\therefore |\vec{p}_j'| = p'$ for $j = 1, \cdots, 4$

All momenta have same magnitude to ensure momentum and energy conservation

$$\therefore \quad p_1' \cdot p_4' = p'^2 + m^2 - p'^2 \cos\theta_s = p'^2(1-\cos\theta_s) + m^2$$

$$t = -(p_1+p_4)^2 = -\underbrace{p_1^2}_{+m^2} - \underbrace{p_4^2}_{m^2} - 2p_1 \cdot p_4.$$

$$= 2m^2 - 2p_1 \cdot p_4$$

$$\therefore \quad \frac{t}{2} = m^2 - \underbrace{p_1 \cdot p_4}_{\substack{\text{Lorentz invariant} \\ \text{scalar } p_1 \cdot p_4}} = m^2 - p_1' \cdot p_4'$$

$$= \cancel{m^2} - p'^2(1-\cos\theta_s) - \cancel{m^2} = -p'^2(1-\cos\theta_s)$$

$$s = -\underbrace{(p_1+p_2)^2}_{\substack{\text{Lorentz invariant} \\ \text{scalar}}} = -(p_1'+p_2')^2 = -\begin{pmatrix} E_1'+E_2' \\ \underbrace{\vec{p}_1' + \vec{p}_2'}_{=0 \ \because \text{ CM frame}} \end{pmatrix}^2$$

$$= -(E_1'+E_2', 0)^2 = (E_1'+E_2')^2$$

$$= (2E_1')^2 = 4(p'^2+m^2)$$

$$\therefore p'^2 = \frac{s}{4} - m^2$$

$$\therefore \quad \frac{t}{2} = -\left(\frac{s}{4} - m^2\right)(1-\cos\theta_s)$$

At high energy $\left|\frac{s}{4}\right| >> |m^2|$

$\therefore$ we have $\frac{s}{4} - m^2 \simeq \frac{s}{4}$

$\therefore \frac{t}{2} = -\frac{s}{4}(1 - \cos\theta_s) \Rightarrow \underline{\underline{2t = -s(1-\cos\theta_s)}}$

~~In CM f~~ Regge limit $s >> 1$, $t < 0$

in CM frame

$\because 2t = -s(1 - \cos\theta_s)$

t is fixed and finite. $(O(1))$

$\because s >> 1 \quad \therefore (1 - \cos\theta_s) << 1$

$\therefore \cos\theta_s \simeq 1 \quad \cancel{\theta_s \simeq 0} \quad \rightarrow 1 - \frac{\theta_s^2}{2} \simeq 1$

$\therefore \underline{\underline{\theta_s << 1}}$ Good!

2) Veneziano Amplitude $\alpha' m^2 = -1$

$$A_V(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

$$\alpha(x) = 1 + \alpha' x$$

$$\Gamma(u) = \int_0^\infty dt\, t^{u-1} e^{-t}$$

Consider Euler beta function $B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$

It has simple poles when ~~u+v~~ u or v is a non-positive integer ✓

There is no double pole because if $\Gamma(u)$ and $\Gamma(v)$ have poles simultaneously ~~have~~ then $\Gamma(u+v)$ also has a pole. ✓

$\Gamma(u)$ has poles at $u = -n$ $(n = 0, 1, 2, \cdots)$

and $\Gamma(u) \sim \frac{1}{u+n} \frac{(-1)^n}{n!}$ near poles.

consider $B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ ✓

$$\bar{B}(u,v) = \sum_{n=0}^{\infty} \frac{1}{v+n} \frac{(-1)^n}{n!} (u-1)\cdots(u-n).$$ ✓

For B as a function of v for fixed u, near the poles of $v=-n$ $\quad B(u,v) \sim \frac{1}{v+n} \frac{(-1)^n}{n!} \frac{\Gamma(u)}{\Gamma(u-n)}$

$\therefore B(u,v) \sim \frac{1}{v+n} \frac{(-1)^n}{n!} (u-1) \cdots (u-n)$

Because $u\Gamma(u) = \Gamma(u+1)$

$\Rightarrow$ we see that $\bar{B}$ has all the poles and same ~~red~~ residue of B.

$\therefore$ $C = B - \bar{B}$, $C(u,v)$ can only be an entire function of v in complex v plane.

$\because$ At poles $B = \bar{B} \Rightarrow C = 0$
Not at poles $B - \bar{B}$ is finite.
At $|v| \to \infty$, $B \to 0$, $\bar{B} \to 0$ $\therefore B - \bar{B} = C \to 0$.
$\therefore$ C is bounded entire function

By Liouville's theorem C is a constant.

good!

look at $u = 1$ $\quad \Gamma(u) = 1$.

$$B(1,v) = \frac{\Gamma(v)}{\Gamma(1+v)} = \frac{1}{v}$$

$$\bar{B} = \frac{1}{v} + \frac{1}{v+1} \cancel{\cos(1)} \cdots$$

$$= \frac{1}{v}$$

$\Rightarrow B - \bar{B} = \frac{1}{v} - \frac{1}{v} = \underline{\underline{0}}$ for all u.

$\therefore \quad B = \bar{B}$

$$\therefore \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} = \sum_{n=0}^{\infty} \frac{1}{v+n} \frac{(-1)^n}{n!} (u-1)\cdots(u-n)$$

$$\therefore A_V(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

$$= \sum_{n=0}^{\infty} \frac{1}{-\alpha(t)+n} \frac{(-1)^n}{n!} (-\alpha(s)-1)(-\alpha(s)-2)\cdots(-\alpha(s)-n)$$

$$= -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)(\alpha(s)+2)\cdots(\alpha(s)+n)}{n!} \frac{1}{\alpha(t)-n}$$

fantastic!

$A_V(s,t)$ can also be written as

$$A_V(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(t)+1)\cdots(\alpha(t)+n)}{n!} \frac{1}{\alpha(s)-n}.$$

keeping t constant and expanding s poles.

Manifestly $\because \; A_V(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))} = A_V(t,s)$

$\therefore$ A_V ~~ex~~ displays Dolan-Horn-Schmid duality

(swapping s and t keeps expression invariant, ~~can~~ sum can be written as sum of either s or t channel poles)

3). $\Gamma(u)$ for large u has the behavior

~~$\Gamma(u) \sim \sqrt{2\pi}\, u^{u-1/2} e^{-u}$~~

$$\Gamma(u) \sim \sqrt{2\pi}\, u^{u-\frac{1}{2}} e^{-u}$$

~~Assume for s large, $\alpha(s)$ is also large~~

~~$\because \alpha(s) = 1+\alpha's$ and~~

$\because \alpha(s) = 1+\alpha' s$ ~~and $\alpha' > 0$~~

$\therefore \alpha(s)$ large as s gets large.

$\because t$ fixed $\therefore \alpha(t) = 1+\alpha' t$ fixed

$$\therefore \Gamma(-\alpha(s)) = \sqrt{2\pi}\,(-\alpha(s))^{-\alpha(s)-\frac{1}{2}} e^{\alpha(s)}$$

$$\Gamma(-\alpha(s)-\alpha(t)) = \sqrt{2\pi}\,(-\alpha(s)-\alpha(t))^{-\alpha(s)-\alpha(t)-\frac{1}{2}} e^{+\alpha(s)+\alpha(t)}$$

~~(Stirling's formula is valid~~

~~$A_V(s,t)$~~ $\therefore$ In Regge limit $s \gg 1$, $t < 0$ fixed.

$$\therefore A_V(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

$$\sim \frac{\cancel{\sqrt{2\pi}}\,(-\alpha(s))^{-\alpha(s)-\frac{1}{2}} e^{\alpha(s)}\,\Gamma(-\alpha(t))}{\cancel{\sqrt{2\pi}}\,(-\alpha(s)-\alpha(t))^{-\alpha(s)-\alpha(t)-\frac{1}{2}} e^{\alpha(s)+\alpha(t)}}$$

$\because |s| \gg |t| \quad \therefore |\alpha(s)| \gg |\alpha(t)|$

correction:

$$\left(\frac{-\alpha(s)}{-\alpha(s)-\alpha(t)}\right)^{-\alpha(s)-\frac{1}{2}} = e^{\alpha(t)} \text{ as } |\alpha(s)| \gg 1$$

$$A_V(s,t) \sim \frac{(-\alpha(s))^{-\alpha(s)-\frac{1}{2}}\,\Gamma(-\alpha(t))}{(-\alpha(s))^{-\alpha(s)-\alpha(t)-\frac{1}{2}}}\,\cancel{e^{-\alpha(t)}}$$

$$\sim \Gamma(-\alpha(t))\,(-\alpha(s))^{\alpha(t)}$$

$\Rightarrow$ good!

s needs to have a small imaginary part because Stirling's formula ~~only~~ works throughout the s plane as long as one keeps away from the positive s axis because ~~a of~~ function $\Gamma(-\alpha(s))$ has poles when $-\alpha(s)$ is negative integers.

to avoid poles, s needs a small imaginary part.

- Av ~~is~~ has many zeros and poles, and the above equation is valid in an average sense. Giving a small ~~is~~ imaginary part is resonable because the resonances are unstable and quantum corrections will give a imaginary part to the position of the poles. good!

4). fixed angle scattering

$$2t = -s(1-\cos\theta_s) \implies t = -\frac{s}{2}(1-\cos\theta_s)$$

high energies $|s| \gg 1 \quad \therefore |t| \gg 1$

$\therefore$ ($\alpha(x) = 1 + \alpha' x \simeq \alpha' x$ for $|x| \gg 1$)

$$\therefore A(s,t) = \frac{\Gamma(-\alpha(s))\,\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$$

$$= \frac{\sqrt{2\pi}\,(-\alpha(s))^{-\alpha(s)-\frac{1}{2}}\,\cancel{e^{+\alpha(s)}}\cdot\cancel{\sqrt{2\pi}}\,(-\alpha(t))^{-\alpha(t)-\frac{1}{2}}\,\cancel{e^{\alpha(t)}}}{\cancel{\sqrt{2\pi}}\,(-\alpha(s)-\alpha(t))^{-\alpha(s)-\alpha(t)-\frac{1}{2}}\,\cancel{e^{\alpha(s)+\alpha(t)}}}$$

$$= \frac{\sqrt{2\pi}\,(-\alpha' s)^{-\alpha' s-\frac{1}{2}}\,\left(-\alpha' s\left(\frac{\cos\theta_s-1}{2}\right)\right)^{-\alpha' s\frac{\cos\theta_s-1}{2}-\frac{1}{2}}}{\left(-\alpha' s-\alpha' s\frac{\cos\theta_s-1}{2}\right)^{-\alpha' s-\alpha' s\frac{\cos\theta_s-1}{2}-\frac{1}{2}}}$$

$$= \sqrt{2\pi}\;\frac{\cancel{(\alpha')^{-\alpha' s-\frac{1}{2}}}\,\cancel{(\alpha')^{-\alpha' s\frac{\cos\theta_s-1}{2}-\frac{1}{2}}}}{\cancel{(\alpha')^{-\alpha' s-\alpha' s\frac{\cos\theta_s-1}{2}}}} \times \frac{\cancel{(-s)^{-\alpha' s-\frac{1}{2}}}\cdot\cancel{(-s)^{-\alpha' s\frac{\cos\theta_s-1}{2}-\frac{1}{2}}}}{\cancel{(-s)^{-\alpha' s-\alpha' s\frac{\cos\theta_s-1}{2}-\frac{1}{2}}}} \times (\;\cdots\;)$$

$$= \sqrt{2\pi} \times (\cdots,) \sim (\cdots,)$$

$$\cancel{\sqrt{2\pi} \times \frac{\cos\theta_s + 1}{2}} \quad \because \ |\alpha' s| \gg \frac{1}{2}$$

$$\therefore \quad \sim \underbrace{\left[\frac{\left(\frac{\cos\theta_s - 1}{2}\right)^{\frac{\cos\theta_s - 1}{2}}}{\left(\frac{\cos\theta_s + 1}{2}\right)^{\frac{\cos\theta_s + 1}{2}}} \right]}_{F(\theta_s)}{}^{\cancel{-\alpha(s)}\ \underbrace{-\alpha' s}_{= -\alpha(s).}}$$

$$\sim \left[F(\theta_s) \right]^{-\alpha(s)}$$

$$\Rightarrow \quad A(s,t) \sim F(\theta_s)^{-\alpha(s)}$$

Falls off exponentially fantastic!

since $\alpha(s) \simeq \alpha' s$ is large and positive.

5) $\alpha_c(x) = 1 + \frac{\alpha' x}{4}$

Virasoro - shapiro amplitude

$$A_{VS}(s,t,u) = \frac{\Gamma(-\alpha_c(s))\,\Gamma(-\alpha_c(t))\,\Gamma(-\alpha_c(u))}{\Gamma(-\alpha_c(s)-\alpha_c(t))\,\Gamma(-\alpha_c(t)-\alpha_c(u))\,\Gamma(-\alpha_c(u)-\alpha_c(s))}$$

with $s+t+u = 4m^2 = -\frac{16}{\alpha'}$ $(*)$

$\because$ the constraint $(*)$, effectively A_{VS} is a function ~~only~~ of s and t, we can ~~exp~~ keep s constant and expand A_{VS} with respect to the poles of t.

$\because$ $\alpha_c(u) = \alpha_c(4m^2 - t - s)\big|_{s\ \text{constant}}$

$\therefore$ when $\alpha_c(4m^2 - t - s)\big|_s = +n$ $(n \in \mathbb{Z}^+)$

$\Gamma(-\alpha_c(u))$ has poles and these are also poles of t, in addition to poles such that $-\alpha_c(t) = +n$ $(n \in \mathbb{Z}^+)$

$\therefore$ we write $A_{VS}(s,t,u)$ as an expansion of t and u channels exchange contributions.

t- channel poles:

$\Gamma(-\alpha_c(t)) \to \infty$

$\alpha_c(t) = n$ $(n = 0, 1, 2, \ldots)$

Near these poles

$$\Gamma(-\alpha_c(t)) \sim \frac{1}{-\alpha_c(t)+n} \frac{(-1)^n}{n!} \qquad \text{Set } -\alpha(t) = -n$$

$$\Rightarrow \frac{\Gamma(-\alpha(s))}{\Gamma(-\alpha(s)-\alpha(t))} = \frac{\Gamma(-\alpha(s))}{\Gamma(-\alpha(s)-n)} = (-\alpha(s)-1)(-\alpha(s)-2) \cdots (-\alpha(s)-n)$$

$$\Rightarrow \frac{\Gamma(-\alpha(u))}{\Gamma(-\alpha(u)-\alpha(t))} = \frac{\Gamma(-\alpha(u))}{\Gamma(-\alpha(u)-n)}$$

$$= (-\alpha(u)-1)(-\alpha(u)-2)\cdots(-\alpha(u)-n)$$

$$\alpha(u) = 1+\frac{\alpha'}{4}u = 1+\frac{\alpha'}{4}\left(\frac{-16}{\alpha'}-t-s\right)$$

$$= -3 - \frac{\alpha'}{4}t - \frac{\alpha'}{4}s$$

$$= -\left(1+\frac{\alpha'}{4}t\right) - \left(1+\frac{\alpha'}{4}s\right) - 1$$

$$= -\alpha(t) - \alpha(s) - 1 = -n-1-\alpha(s)$$

$$\therefore \quad -(\alpha(u)+\alpha(t)+\alpha(s)) = 1$$

$$(\Rightarrow -\alpha(u) = n+1+\alpha(s))$$

$$= (-\alpha(u)-1)\cdots(-\alpha(u)-n)$$

$$= (\alpha(s)+n)(\alpha(s)+n-1)\cdots(\alpha(s)+1)$$

$$\Rightarrow \frac{1}{\Gamma(-\alpha(s)-\alpha(u))} = \frac{1}{\Gamma(1+\alpha(t))} = \frac{1}{\Gamma(1+n)}$$

$\because n = (0, 1, 2, \cdots) \quad \therefore \quad n+1 = 1, 2, 3, \cdots$

$\cancel{\Gamma(1), \Gamma(2)}$

$\therefore \quad = \frac{1}{n!}$

$\Rightarrow$ At pole of the $\Gamma(-\alpha(t)) \Rightarrow \alpha(t) = n$

$$A_{VS}(s,t,u) \sim \frac{1}{-\alpha(t)+n} \frac{(-1)^n}{n!} \frac{1}{n!} (-\alpha(s)-1)(-\alpha(s)-2)\cdots$$

$$(-\alpha(s)-n) \times (\alpha(s)+n)(\alpha(s)+n-1)\cdots(\alpha(s)+1)$$

$$\sim \frac{1}{-\alpha(t)+n} \left(\frac{1}{n!}\right)^2 (\alpha(s)+1)^2(\alpha(s)+2)^2\cdots(\alpha(s)+n)^2$$

Similarly ~~for~~ for u - channel poles

$$A_{VS} \sim \frac{1}{-\alpha(u)+n} \left(\frac{1}{n!}\right)^2 (\alpha(s)+1)^2(\alpha(s)+2)^2\cdots(\alpha(s)+n)^2$$

$\therefore$ $A_{VS}(s,t,u) =$

$$-\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)^2(\alpha(s)+2)^2\cdots(\alpha(s)+n)^2}{(n!)^2}\left(\frac{1}{\alpha(t)-n} + \frac{1}{\alpha(u)-n}\right)$$

where $\alpha(u) + \alpha(t) + \alpha(s) = -1$ good!

6) - In Regge limit $s \gg 1$ $t<0$ fixed.

$$\alpha(s) = 1 + \frac{\alpha'}{4}s \simeq \frac{\alpha'}{4}s.$$

$\therefore \alpha(s) + n \simeq \alpha(s)$ for finite n.

$$\therefore A_{VS}(s,t,u) \simeq -\sum_{n=0}^{\infty} \frac{(\alpha(s))^{2n}}{(n!)^2} \left(\frac{1}{\alpha(t)-n} + \frac{1}{\alpha(u)-n} \right).$$

- high energy fixed angle.

$2t = -s(1-\cos\theta_s)$ $\therefore$ s/t fixed

as $s, t \to \infty$

using $\Gamma(x) \sim \exp(x \ln x)$

(Stirling formula $\Gamma(x) \sim \sqrt{2\pi}\, x^{x-\frac{1}{2}} e^{-x}$. $\sim x^x e^{-x}$.

$\sim e^{x\ln x} e^{-x} \sim e^{x\ln x - x}$

as $x \to \infty$ $\sim e^{x \ln x}$).

$$\therefore A(s,t,u) = \frac{\Gamma(-1-\frac{\alpha' s}{4})\,\Gamma(-1-\frac{\alpha' t}{4})\,\Gamma(-1-\frac{\alpha' u}{4})}{\Gamma(2+\frac{\alpha' s}{4})\,\Gamma(2+\frac{\alpha' t}{4})\,\Gamma(2+\frac{\alpha' u}{4})}$$

$$\sim \frac{\Gamma(-\frac{\alpha' s}{4})\,\Gamma(-\frac{\alpha' t}{4})\,\Gamma(-\frac{\alpha' u}{4})}{\Gamma(\frac{\alpha' s}{4})\,\Gamma(\frac{\alpha' t}{4})\,\Gamma(\frac{\alpha' u}{4})}$$

$$\Gamma(x) \sim x^x \sim e^{x\ln x} \sim e^{x\ln|x|}$$

$$\Gamma(-x) \sim (-x)^{(-x)} \sim e^{-x\ln|x|}(-1)^{-x} \sim e^{-x\ln|x|}$$

$$\therefore \quad A(s,t,u) \sim \frac{e^{-\frac{\alpha'}{4}s\ln|s|}\, e^{-\frac{\alpha'}{4}t\ln|t|}\, e^{-\frac{\alpha'}{4}u\ln|u|}}{e^{\frac{\alpha'}{4}s\ln|s|}\, e^{\frac{\alpha'}{4}t\ln|t|}\, e^{\frac{\alpha'}{4}u\ln|u|}}$$

$$\sim \exp\left(-\frac{\alpha'}{2}\left(s\ln|s| + t\ln|t| + u\ln|u|\right)\right).$$

$\Rightarrow$ Amplitude decays exponentially

good!

Q2

1) Nambu-Goto Lagrangian

$$\mathcal{L}_{NG} = -T\sqrt{-h} = \cancel{\mathcal{L}(x)}$$

$$= -T\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2} = \mathcal{L}(\dot{X}^\mu, X'^\mu)$$

where $\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$, $X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$

and $\tau = \xi^1$, $\sigma = \xi^2$

The action $S = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \; \mathcal{L}(\dot{X}^\mu, X'^\mu)$

$(\sigma \in [0, \pi])$

Equation of motion $\Rightarrow$ $\delta S = 0$

$$\Rightarrow \quad 0 = \delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu}}_{K^\tau_\mu} \delta \dot{X}^\mu + \underbrace{\frac{\partial \mathcal{L}}{\partial X'^\mu}}_{K^\sigma_\mu} \delta X'^\mu \right)$$

$$\left(\delta \dot{X}^\mu = \delta\left(\frac{\partial X^\mu}{\partial \tau}\right) = \frac{\partial(\delta X^\mu)}{\partial \tau} , \quad \delta X'^\mu = \frac{\partial(\delta X^\mu)}{\partial \sigma} \right)$$

$$= \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\frac{\partial \delta X^\mu}{\partial \tau} K^\tau_\mu + \frac{\partial \delta X^\mu}{\partial \sigma} K^\sigma_\mu \right]$$

$$= \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma \left[\frac{\partial}{\partial \tau}(\delta X^\mu K^\tau_\mu) + \frac{\partial}{\partial \sigma}(\delta X^\mu K^\sigma_\mu) - \delta X^\mu \left(\frac{\partial K^\tau_\mu}{\partial \tau} + \frac{\partial K^\sigma_\mu}{\partial \sigma} \right) \right].$$

$$= \underbrace{\int_0^\pi d\sigma \left[\delta X^\mu K_\mu^\tau\right]_{\tau_i}^{\tau_f}}_{①} + \underbrace{\int_{\tau_i}^{\tau_f} d\tau \left[\delta X^\mu K_\mu^\sigma\right]_0^\pi}_{②}$$

$$\underbrace{- \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma\, \delta X^\mu \left(\frac{\partial K_\mu^\tau}{\partial \tau} + \frac{\partial K_\mu^\sigma}{\partial \sigma} \right)}_{③}$$

① $= 0$ $\because$ $\delta X^\mu(\tau_f, \sigma) = \delta X^\mu(\tau_i, \sigma) = 0$
(initial and final states of the string is not varied).

② $= 0$ $\because$ $K_\mu^\sigma(\tau, 0) = K_\mu^\sigma(\tau, \pi) = 0$
By free-endpoint boundary conditions for open strings. And for closed strings, $\sigma = 0$ and $\sigma = \pi$ are the same point so ② also vanishes.

$\Rightarrow$ ③ $= 0$ for every δX^μ $\Rightarrow$

equation of motion is $\dfrac{\partial K_\mu^\tau}{\partial \tau} + \dfrac{\partial K_\mu^\sigma}{\partial \sigma} = 0$

$\Rightarrow \partial_i K_\mu^i = 0$ $\square$ (e.o.m)

$$P^\mu(\tau) = \int_0^\pi d\sigma\, K^{\mu\tau}(\sigma, \tau)$$

then $\dfrac{dP^\mu(\tau)}{d\tau} = \displaystyle\int_0^\pi d\sigma \frac{\partial K^{\mu\tau}}{\partial \tau} = - \int_0^\pi d\sigma \frac{\partial K^{\mu\sigma}}{\partial \sigma}$

$$= -K^{\mu\sigma} \Big|_0^\pi$$

For closed string $\sigma=\pi$ and $\sigma=0$ are same point so $K^{\mu\sigma}(\pi)=K^{\mu\sigma}(0)$ ✓

For open string apply free-endpoint boundary condition

$$K^{\mu\sigma}(\pi)=K^{\mu\sigma}(0)=0.$$ ✓

In both cases $K^{\mu\sigma}\Big|_0^\pi=0$ ✓

✓

$$\therefore \frac{dP^\mu(\tau)}{d\tau}=0$$

And by reparameterisation invariance of the action S we can choose a gauge such that $t=\tau$ where $t=x^0$ is the Minkowski time coordinate

$$\Rightarrow \frac{dP^\mu(\tau)}{dt}=0 \quad \text{and} \quad P^\mu(\tau) \text{ is conserved.}$$

The density is given by $K^{\mu,\tau}$

$P^\mu(\tau)$ is the σ density of spacetime momentum carried by the string good!

2) $$M^{\mu\nu}=\int_0^\pi d\sigma\,\underbrace{(x^\mu K^{\nu\tau}-x^\nu K^{\mu\tau})}_{N^{\mu\nu\tau}}=\int_0^\pi d\sigma\, N^{\mu\nu\tau}$$

define $N^{\mu\nu\sigma}=x^\mu K^{\nu\sigma}-x^\nu K^{\mu\sigma}$ ✓

compute $$\frac{\partial N^{\mu\nu\tau}}{\partial\tau}=\dot{x}^\mu K^{\nu\tau}+x^\mu\frac{\partial K^{\nu\tau}}{\partial\tau}-\dot{x}^\nu K^{\mu\tau}-x^\nu\frac{\partial K^{\mu\tau}}{\partial\tau}$$

$\because K^{\mu\tau} = \frac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}}$ //

In conformal gauge $g_{\mu\nu} = e^{2\phi(\sigma,\tau)} \eta_{\mu\nu}$

$\eta = \text{diag}(-1, 1, 1, 1)$

Nambu-Goto ~~Lagrangian~~ action

$$S = -T \int d\tau d\sigma \sqrt{-h}$$ is ~~cat~~ classically

equivalent to the Polyakov action

$$S = -\frac{T}{2} \int d\tau d\sigma \left(\sqrt{-h}\, h^{ab} \partial_a X \partial_b X \right).$$

and in conformal gauge this is equal to

$$S = -\frac{T}{2} \int d\tau d\sigma\, \eta^{ab} \partial_a X \partial_b X$$

$$\left(\because\ h_{ab} = e^{2\phi} \eta_{ab} \Rightarrow h = -e^{4\phi}\ ;\ h^{ab} = e^{-2\phi} \eta^{ab} \right.$$
$$\left. \sqrt{-h}\, h^{ab} = \underbrace{\sqrt{-\eta}}_{1}\, \eta^{ab} \cancel{e^{2\phi}} \cancel{e^{-2\phi}} = \eta^{ab} \right)$$

$$S = -\frac{T}{2} \int d\tau d\sigma\, \partial_a X \cdot \partial^a X. \cancel{= -\frac{T}{2} \int d\tau d\sigma}$$

Polyakov lagrangian $\mathcal{L} = -\frac{T}{2} \partial_a X \cdot \partial^a X.$

$(a = \tau, \sigma)$ /

$$\mathcal{L} = -\frac{T}{2}\,\partial_a X \cdot \partial^a X$$

$$= \frac{T}{2}\left(\dot{X}^2 - X'^2\right) \qquad \left(\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}, \quad X'^\mu = \frac{\partial X^\mu}{\partial \sigma} \right)$$

$$\therefore \quad K^\tau_\mu = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T \dot{X}_\mu$$

$$\rightarrow K^{\mu\tau} = T\dot{X}^\mu, \quad K^{\mu\sigma} = -T X'^\mu$$

$$\therefore \quad \frac{\partial N^{\mu\nu\tau}}{\partial \tau} = \cancel{T\dot{X}^\mu \dot{X}^\nu} + X^\mu \frac{\partial K^{\nu\tau}}{\partial \tau} - \cancel{T\dot{X}^\nu \dot{X}^\mu} - X^\nu \frac{\partial K^{\mu\tau}}{\partial \tau}$$

$$\frac{\partial N^{\mu\nu\sigma}}{\partial \sigma} = X'^\mu K^{\nu\sigma} + X^\mu \frac{\partial K^{\nu\sigma}}{\partial \sigma} - X'^\nu K^{\mu\sigma} - X^\nu \frac{\partial K^{\mu\sigma}}{\partial \sigma} \quad \checkmark$$

$$= -\cancel{T X'^\nu X'^\mu} + X^\mu \frac{\partial K^{\nu\sigma}}{\partial \sigma} + \cancel{T X'^\mu X'^\nu} - X^\nu \frac{\partial K^{\mu\sigma}}{\partial \sigma}$$

$$\Rightarrow \frac{\partial N^{\mu\nu\sigma}}{\partial \sigma} + \frac{\partial N^{\mu\nu\tau}}{\partial \tau} \quad \checkmark$$

$$= X^\mu \underbrace{\left(\frac{\partial K^{\nu\tau}}{\partial \tau} + \frac{\partial K^{\nu\sigma}}{\partial \sigma} \right)}_{0} - X^\nu \underbrace{\left(\frac{\partial K^{\mu\tau}}{\partial \tau} + \frac{\partial K^{\mu\sigma}}{\partial \sigma} \right)}_{0}$$

$$= 0 \quad \checkmark$$

good!

$$M^{\mu\nu} = \int_0^{\pi} d\sigma \, N^{\mu\nu\tau}$$

$$\therefore \frac{dM^{\mu\nu}}{d\tau} = \int_0^{\pi} d\sigma \, \frac{\partial N^{\mu\nu\tau}}{\partial \tau}$$

$$= \int_0^{\pi} d\sigma \left(-\frac{\partial N^{\mu\nu\sigma}}{\partial \sigma} \right).$$

$$= \left[-N^{\mu\nu\sigma} \right]_0^{\pi}$$

~~For~~ closed string $N(\pi) = N(0)$

$\therefore$ this is 0 ✓

For open ~~str.~~ string

$K^{\mu\tau}(\pi) = K^{\mu\tau}(0) = K^{\nu\tau}(\pi) = K^{\nu\tau}(0) = 0$

$\Rightarrow$ ~~this~~ $\cancel{\frac{dM^{\mu\nu}}{d\tau} = 0}$ $N^{\mu\nu\sigma}(\pi) = N^{\mu\nu\sigma}(0) = 0$

✓

In both cases $\frac{dM^{\mu\nu}}{d\tau} = 0$

choose $\tau = t$ gives $\underline{\underline{\frac{dM^{\mu\nu}}{dt} = 0}}$

✓

$\Rightarrow$ $M^{\mu\nu}$ is conserved.

3) From lecture notes,

For the Polyakov action in conformal gauge

equation of motion of the metric is

$$T_{ab} = -\frac{2}{T}\frac{1}{\sqrt{-h}}\frac{\partial S}{\partial h^{ab}} = 0$$

$$\Rightarrow \quad T_{\tau\tau} = T_{\sigma\sigma} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0 \quad ①$$

$$T_{\sigma\tau} = T_{\tau\sigma} = \dot{X}\cdot X' = 0 \quad ②$$

hold at every point of the ~~w~~ world-sheet.

Neumann boundary conditions:

K ~~...~~ at boundary,

$$K^{\mu\sigma} = X'^{\mu} = 0 \quad ③$$

② means that $\because$ X'^{μ} is tangent to

the string (~~...~~ $X'^{\mu} = \frac{\partial X^{\mu}}{\partial \sigma}$) $\therefore$ $\dot{X}$ is

~~perpendi~~

the velocity vector of the string

③ and ① gives $\dot{X}^2 = 0$ at end points

$\therefore$ At end points, the Y - velocity

good!

vector of the string is null)

$\Rightarrow$ end point moves at speed of light.

4) $$\left.\begin{aligned} X^0 &= \tfrac{1}{2}\left(p+\tfrac{a^2}{p}\right)n\tau \\ X^1 &= \tfrac{1}{2}\left(p-\tfrac{a^2}{p}\right)n\tau \\ X^2 &= a\cos(n\sigma)\cos(n\tau) \\ X^3 &= a\cos(n\sigma)\sin(n\tau) \end{aligned}\right\} \quad (*)$$

Equations of motion in conformal gauge

$$\partial_i K^i_\mu = 0 \rightarrow \partial_i K^{i\mu} = 0 \Rightarrow \frac{\partial K^{\mu\tau}}{\partial \tau} + \frac{\partial K^{\mu\sigma}}{\partial \sigma} = 0$$

$$K^{\mu\tau} = \dot{X}^\mu \qquad K^{\mu\sigma} = -X'^\mu \quad \text{(conformal gauge)}$$

$$\therefore \quad \underline{\underline{\ddot{X}^\mu - X''^\mu = 0}} \qquad \text{(E.O.M)}$$

For $\mu = 0$

$$\dot{X}^0 = \tfrac{1}{2}\left(p+\tfrac{a^2}{p}\right)n \qquad \ddot{X}^0 = 0$$

$$X''^0 = 0 \Rightarrow \ddot{X}^0 - X''^0 = 0 \quad ✓$$

$\mu = 1$ $\qquad \ddot{X}^1 = 0 \qquad X''^1 = 0 \qquad \therefore \ddot{X}^1 - X''^1 = 0$ ✓

$\mu = 2$ $$\ddot{X}^2 = -n^2 a\cos(n\sigma)\cos(n\tau)$$
$$X''^2 = -n^2 a\cos(n\sigma)\cos(n\tau)$$

$$\Rightarrow \ddot{X}^2 - X''^2 = 0 \quad ✓$$

$\mu = 3$ $$\ddot{X}^3 = -n^2 a\cos(n\sigma)\sin(n\tau)$$
$$X''^3 = -n^2 a\cos(n\sigma)\sin(n\tau)$$

$$\Rightarrow \ddot{X}^3 - X''^3 = 0 \quad ✓$$

$\therefore$ (*) is a solution in conformal gauge.

This string is a straight open string ~~rotates~~ rotating rigidly about its ~~mid point~~ mid point (centre of mass), which moves along a straight line in the x-direction.

The string rotates in the y-z plane

~~A~~ a solution that centre of mass is stationary:

correction

set $a=p$

$\partial_\tau X^1=0$

$x^0 = pn\tau$

$$X^0 = \tau$$
$$X^1 = 0$$
$$X^2 = a\cos(n\sigma)\cos(n\tau)$$
$$X^3 = a\cos(n\sigma)\sin(n\tau)$$

Calculating the ~~oo~~ only non-vanishing angular momentum component M^{23} ~~M^{32}~~

$$M^{23} = \int_0^{\sigma_1} d\sigma \; X^2 \mathcal{P}^{3\tau} - X^3 \mathcal{P}^{2\tau} \qquad (\sigma_1 = \pi)$$

$$\mathcal{P}^{2\tau} = T\dot{X}^2 = ~~-an\sin(n\sigma)\cos(n\ldots~~ - an\cos(n\sigma)\sin(n\tau)\,T$$

$$\mathcal{P}^{3\tau} = T\dot{X}^3 = an\cos(n\sigma)\cos(n\tau)\,T$$

$\therefore \quad X^2 \dot{X}^3 - X^3 \dot{X}^2$

$$= \cancel{\cdot}\, Ta^2 n \cos^2(n\sigma)\,\big(\underbrace{\cos^2(n\tau) + \sin^2(n\tau)}_{1}\big)$$

$$= Ta^2 n \cos^2(n\sigma) \quad \checkmark$$

$$M^{23} = \int_0^{\sigma_1} d\sigma\, Ta^2 n \cos^2(n\sigma)$$

$$= Ta^2 n \int_0^{\sigma_1} \cancel{d\sigma}\, \cos^2(n\sigma)\, d\sigma$$

$$\cos^2(n\sigma) = \frac{1}{2} + \frac{1}{2}\cos(2n\sigma)$$

n = positive integer

$$M^{23} = \frac{Ta^2 n \sigma_1}{2}\Big|_{\sigma_1 = \pi} + Ta^2 n \int_0^{\pi} \underbrace{\cos(2n\sigma)\, d\sigma}_{\frac{1}{2n}(\sin(2n\pi) - \sin(2n(0))) = 0}$$

$$\therefore M^{23} = \frac{Ta^2 n \pi}{2} \quad \checkmark$$

$$= \cancel{\tfrac{Tn\pi}{2a}}\left(\frac{1}{2}Tn\pi\right)(a^2)$$

$\therefore$

$a \sim$ spacetime energy , ~~M^{23}~~

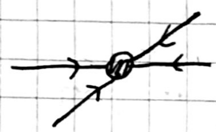
$M^{23} \sim$ angular momentum

$\therefore$

$$(\text{Angular momentum}) \underset{\text{good!}}{\sim} (\text{space time energy})^2$$

String Theory I class I

Q1



$P_1 = (E, 0, 0, p)$
$P_2 = (E, 0, 0, -p)$
$P_3 = (-E, p\sin\theta_s, 0, p\cos\theta_s)$
$P_4 = (-E, -p\sin\theta_s, 0, -p\cos\theta_s)$

$s = -(P_1+P_2)^2$
$t = -(P_1+P_4)^2$
$u = -(P_1+P_3)^2$
$-E^2+p^2 = 1$ for tachyons

$s = 4E^2$, $t = -p^2(2-2\cos\theta_s)$
$= -(4+s)\sin^2\frac{\theta_s}{2}$

$\rightarrow \sin^2\frac{\theta_s}{2} = \frac{-t}{4+s}$, $\cos^2\frac{\theta_s}{2} = -\frac{u}{4+s}$, Regge limit $\rightarrow\leftarrow \Rightarrow \leftarrow\rightarrow$

Fixed angle $s \gg 1$, $t \ll -1$, $\frac{-t}{s}$ fixed to $\sin^2\frac{\theta_s}{2}$

Veneziano : $A_V(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$ $\alpha(x) = 1+\alpha' x$

as a function of t, poles at $-\alpha(t) = 0, -1, -2, \dots$

$A_{poles}(s,t) \text{ “=” } \sum_{n=0,1,2,\dots} \frac{(-1)^n\Gamma(-\alpha(s))}{\Gamma(-\alpha(s)-n)} \frac{1}{n!(-\alpha(t)+n)}$

$\left(\mathrm{Res}_{z=-n}\Gamma(z)\right) = \frac{(-1)^n}{n!}$
$\Gamma(x) = (x-1)\Gamma(x-1)$

$= \sum_{n=0}^{\infty} \frac{(-1)^n(-\alpha(s)-1)\cdots(-\alpha(s)-n)}{n!(-\alpha(t)+n)}$

(1) $A_{poles}(s,t)$ exists (for $\alpha(t) \neq (n=0,1,\dots)$).

$\Rightarrow$ $A_{poles}(s,t) = A_V(s,t) + \underbrace{\text{entire function of } t}_{f(t)}$.

(2) $f(t) = \cancel{\ } \text{const.}$ $\rightarrow$ $|f(t)| \underset{|t|\gg 1}{\sim} |t|^{\#\geq 1}$

$A_V(s,t)$ & A_{poles} $\rightarrow$ const at $|t| \gg 1$.

at const t $\sum_{n=0}^{\infty} C_n$ $C_n = \frac{(-1)^n\Gamma(-\alpha(s))}{\Gamma(-\alpha(s)-n)\Gamma(n+1)(-\alpha(t)+n)}$

(duplication formula $\Gamma(x-n) = (-1)^{n-1}\frac{\Gamma(-x)\Gamma(1+x)}{\Gamma(n+1-x)}$)

$C_n = g(s)\left(\frac{\Gamma(\alpha(s)+1+n)}{\Gamma(1+n)(\alpha(t)+n)}\right)$ Stirling : $\Gamma(x) \simeq (\frac{x}{e})^x\sqrt{\frac{2\pi}{x}} \sim e^{x\log x - x}$

$\rightarrow C_n \simeq \tilde{g}(s)\frac{n^{\alpha(s)}}{(\alpha(t)+n)}$ $\therefore$ converges for $\mathrm{Re}(\alpha(s)) < 0$

$\sim \frac{1}{n}$ harmonic series maximally not convergent.

exists a unique analytic continuation outside this range.

Regge limit $s \gg 1$, t fixed :

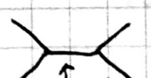
Stirling true outside this wedge with arbitrary small angle.

$\Gamma(-\alpha(s)) = \Gamma(-1-\alpha' s)$ $s \gg 1$ real.

Problem : poles at $\alpha' s = -1, 0, 1, 2, 3, \dots$

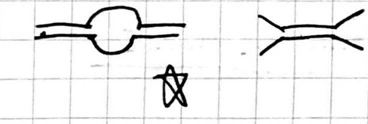
Physics : this come from exchange of massive particles of $m^2 = n = -1, 0, 1, 2, \dots$

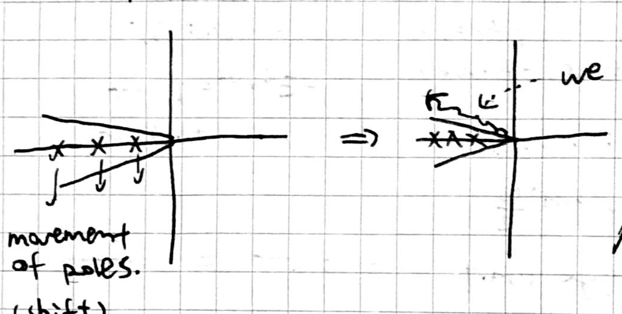
$\leftarrow$ These should be unstable particles.



massive $\frac{1}{s-m^2}$

(loop amplitudes like ✡ give imaginary part to the pole parts of unstable particles)





excuse to just use the

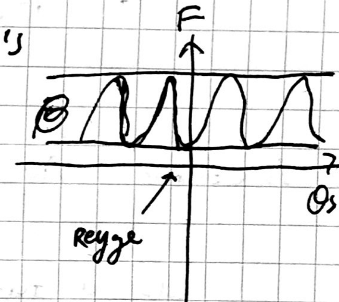
$$\Gamma(x) \simeq \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}} \quad \text{when } x \ll -1 \quad \sim e^{x\log x - x}$$

$$A_V(s,t) \underset{t \text{ fixed}}{\overset{s\gg 1}{\sim}} \Gamma(-\alpha(t))\,(-\alpha' s)^{1+\alpha' t}$$

$$F(\theta_s) = \frac{1}{\sin^{2\sin^2}\ \cos^{2\cos^2}}$$

$$A_V(s,t) \overset{s\gg 1}{\sim} \propto F(\theta_s)^{-\alpha' s}$$

~~t fixed~~ $t \ll -1$, $-\frac{t}{s} = \sin^2\frac{\theta_s}{2}$



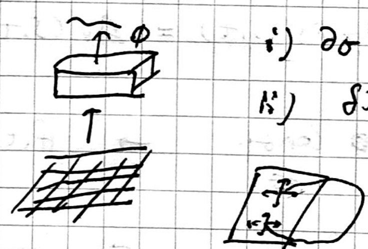
$$A_{VS} = \sum_{n=0}^{\infty} \frac{\left((\alpha(s)+n)\cdots(\alpha(s)+1)\right)^2}{(n!)^2}\left\{\frac{1}{-\alpha(t)+n} + \frac{1}{1+\alpha(t)+\alpha(s)+n}\right\}$$

Regge: $A_{VS}(s,t,u) \simeq \dfrac{\Gamma(-\alpha(t))}{\Gamma(1+\alpha(t))}\left(-\dfrac{\alpha' s}{4}\right)^{2+\frac{\alpha' t}{2}}$

fixed angle: $A_V(s,t,u) \sim F(\theta_s)^{-\frac{\alpha' s}{2}}$ same θ_s as before.

$\boxed{Q2}$ $\quad \mathcal{L} = -T\sqrt{-h}$, $h = \left((\partial_\tau X)^2(\partial_\sigma X)^2 - (\partial_\tau X\cdot\partial_\sigma X)^2\right)$

$\delta S = \text{B.T} + \text{eom}$



i) $\partial_\sigma X(\sigma,\tau)\big|_{\sigma=0,\pi} = 0$.

ii) $\delta X\big|_{\sigma=0,\pi} = 0$.

$$K^\alpha_\mu = \frac{\delta\mathcal{L}}{\delta\partial_\alpha X^\mu}, \quad \underbrace{\partial_\alpha K^\alpha_\mu = \frac{\delta\mathcal{L}}{\delta X^\mu}}_{\text{By eom}} = 0$$

$$P(\tau) = \int_0^\pi d\sigma\, K^\tau_\mu, \quad \partial_\tau P_\mu \overset{?}{=} 0$$

$$= \int_0^\pi d\sigma\,\partial_\tau K^\tau_\mu = \int_\pi^0 d\sigma\,\partial_\sigma(K^\sigma_\mu) = K^\sigma_\mu\Big|_\pi^0$$

closed string - periodic → vanishes, open string ∵ B.C ∴ vanishes.

Conformal gauge $(\sigma,\tau) \to (\sigma'(\sigma,\tau), \tau'(\sigma,\tau))$

$$\begin{pmatrix} \partial_{\tau'}X\cdot\partial_{\tau'}X & \partial_{\tau'}X\cdot\partial_{\sigma'}X \\ \partial_{\sigma'}X\cdot\partial_{\tau'}X & \partial_{\sigma'}X\cdot\partial_{\sigma'}X \end{pmatrix} = e^{2\omega(\sigma,\tau)}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

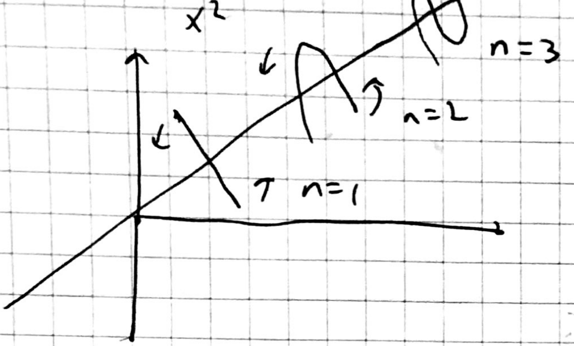
$$M^{\mu\nu} = \int_0^\pi d\sigma\,\left[X^\mu K^{\nu\tau} - X^\nu K^{\mu\tau}\right] \quad \text{(Lorentz symmetry)}$$

→ $\partial_\tau X\cdot\partial_\tau X + \partial_\sigma X\cdot\partial_\sigma X = 0$ At boundary $\partial_\sigma X = 0$.

$\therefore\ \partial_\tau X\cdot\partial_\tau X = 0$

↳ Tangent vector of trajector of end points.

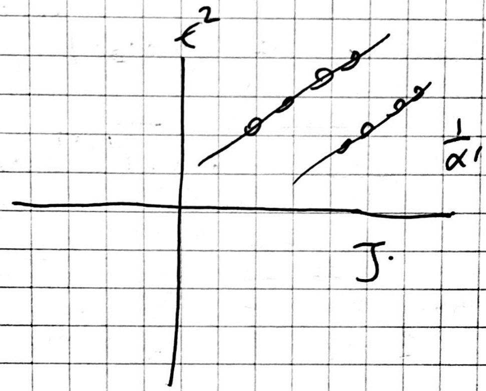
$U_\mu U^\mu = 0$ → null vector → speed of light

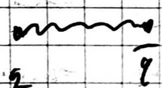


p^0, M^{23} — $(p^0)^2$, M^{23}

$\wr$ M^2 $\qquad \wr$ ℓ.

$$\boxed{\frac{\varepsilon^2}{\ell} = \frac{n}{\alpha'}}$$





Conformal gauge $\quad K_\mu{}^\alpha = T\partial^\alpha X_\mu \quad$ E.o.m: $\partial_\alpha \partial^\alpha X_\mu = 0$

$$P_\mu \sim (P_0)^2 = E$$

$$M^{23} \sim \ell$$

$$\dot{x}^2 = -e^{2\omega} \qquad x'^2 = e^{2\omega} \qquad \dot{x}\cdot x' = 0.$$

$$K_\mu{}^\tau = -T \frac{(\dot{x}\cdot x')x_\mu' - (x')^2 \dot{x}_\mu}{\sqrt{(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2}} = -T \frac{(0) - e^{2\omega}\dot{x}_\mu}{\sqrt{-(-e^{2\omega})e^{2\omega}}}$$

$$= -T \frac{-e^{2\omega}}{e^{2\omega}} \dot{x}_\mu = T\dot{x}_\mu$$

$$\boxed{\begin{aligned} x' &= \partial_\sigma X \\ \dot{x} &= \partial_\tau X \end{aligned}}$$

similarly $\quad K_\mu{}^\sigma = -T x_\mu'$

e.o.m is the same for Polyakov action & Nambu-Goto action.

$$\frac{(-\alpha(s))^{-\alpha(s)-\frac{1}{2}}}{(-\alpha(s)-\alpha(t))^{-\alpha(s)-\alpha(t)-\frac{1}{2}}}$$

$$= \frac{(-\alpha(s))^{-\alpha(s)-\frac{1}{2}}}{(-\alpha(s))^{(-\alpha(s)-\alpha(t)-\frac{1}{2})}} \times \frac{1}{\left(1+\frac{\alpha(t)}{\alpha(s)}\right)^{\underbrace{-\alpha(s)-\alpha(t)-\frac{1}{2}}_{\sim -\alpha(s)}}}$$

$$= (-\alpha(s))^{\alpha(t)} \times \left(1+\frac{\alpha(t)}{\alpha(s)}\right)^{\alpha(s)}.$$

$$= (-\alpha(s))^{\alpha(t)} \times \underbrace{\left(\underbrace{\left(1+\frac{1}{\left(\frac{\alpha(s)}{\alpha(t)}\right)}\right)^{\frac{\alpha(s)}{\alpha(t)}}}_{\to e \text{ as } \frac{\alpha(s)}{\alpha(t)} \to \infty.}\right)^{\alpha(t)}}_{\sim e^{\alpha(t)}}$$

cancels with that extra $e^{-\alpha(t)}$