String Theory I: Problem Sheet 3

Hilary Term 2018

[Last update: 18:06 on Wednesday 14th February, 2018]

1. Conformal properties of local operators

Recall that we called a boundary local operator $\mathcal{O}(\tau)$ a *primary of dimension h* if its commutation relations with the Virasoro charges take the form

$$[L_m, \mathcal{O}(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + mh\right)\mathcal{O}(\tau)\ .$$

On the second problem sheet you verified the closed-string version of the fact that the scalar fields $X^\mu(\tau)$ transform as primaries of dimension $(0,0)$, and in lecture it was stated that the boundary operator $X^\mu(\tau)$ transforms as a primary of dimension $h=0$.

1. Show that the first derivative of the boundary scalar field, $\partial_\tau X^\mu(\tau)$, is a primary field of dimension $h=1$.
2. Show that the *second derivative* of the boundary scalar field, $\partial_\tau^2 X^\mu(\tau)$, is not a primary field of any dimension.
3. The vertex operator associated to absorption of an open string tachyon with spacetime momentum k is given as follows:

$$V(k;\tau) =:\ \exp\left(k\cdot\sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}e^{in\tau}\right)e^{ik\cdot(x+p\,\tau)}\exp\left(-k\cdot\sum_{n=1}^{\infty}\frac{\alpha_n}{n}e^{-in\tau}\right):$$

   Show that this is a primary operator of dimension $h = k\cdot k/2$.

2. The dilaton in OCQ

We saw in lecture that at the first excited level of the closed string, there are $(D-2)^2$ physical states that organize into a graviton, an anti-symmetric two-form, and a scalar under space-time Poincaré transformations. We did not, however, write down an expression for the physical state corresponding to the scalar degree of freedom (the *dilaton*, usually denoted by ϕ).

1. Your first guess for the dilaton state in the closed-string Hilbert space would be to write

$$|\phi;p\rangle \stackrel{?}{=} (\alpha_{-1}\cdot\tilde{\alpha}_{-1})\,|0;p\rangle\ ,$$

   Check that this state does not satisfy the L_{+1} physical state condition. Find a way to improve this state to render it physical. Show that the resulting state that you write down represents a space-time scalar particle.

2. Find a vertex operator $V_\phi(\tau,\sigma)$ for the dilaton. Check that it is a primary operator of dimension $(1,1)$. Check that it satisfies the operator/state relation,[1]

$$\lim_{\tau\to i\infty} e^{-4i\tau}V_\phi(\tau,\sigma)|0;0\rangle = |\phi;p\rangle\ .$$

[1] Note that this is the operator/state relation relevant for the (slightly unusual!) GSW conventions for the closed string, where world-sheet time translations are generated by $H = 2(L_0+\tilde{L}_0)$. It would be slightly more standard to define $\tau_{\rm new} = 2\tau_{\rm old}$, in terms of which we would have

$$\lim_{\tau\to i\infty} e^{-2i\tau_{\rm new}}V_\phi(\tau_{\rm new},\sigma)|0;0\rangle = |\phi;p\rangle\ .$$

3. String three-point couplings

In lecture, we will show/have showed that the three-point coupling of open string tachyons takes the extremely simple form (in conventions where all momenta are incoming)

$$\mathcal{A}_3(k_1, k_2, k_3) = g_o \langle 0; k_1 | V(k_2, \tau) | 0; k_3 \rangle = g_o \, \delta(k_1 + k_2 + k_3) \ ,$$

where g_o is the open string coupling constant.

1. Compute the three-point coupling of two open string tachyons with one photon.
2. Compute the three-point coupling of one open string tachyon with two photons.
3. Verify that the couplings respect gauge invariance of the photon.

4. Products of vertex operators and the Veneziano amplitude

In the context of string scattering amplitudes for four or more particles, one necessarily has to deal with products of vertex operators. This made easier by being able to normal order the products.

1. Prove the formula

$$V(k_1, \tau_1) V(k_2, \tau_2) =: V(k_1, \tau_1) V(k_2, \tau_2) : \left(e^{i\tau_1} - e^{i\tau_2}\right)^{k_1 \cdot k_2}$$

   where the colons around the product mean that the entire product is defined according to creation/annihilation normal ordering conventions. You should make sure your conventions for the arrangement of the zero modes x^μ and p^μ are such that the normal ordering rule takes this simple form. If required, you may give τ_1 and τ_2 small imaginary components and make whatever assumptions are necessary about their relative magnitude.

2. Use this result to derive the integral expression for the Veneziano amplitude for the scattering of four open string tachons,

$$V(k_1, k_2, k_3, k_4) = g_o^2 \int_{-\infty}^{0} d\mathbf{t} \ \langle 0; k_1 | V(k_2, 0) V(k_3, \tau = -i\mathbf{t}) | 0; k_4 \rangle \ .$$

Please send comments and corrections to `christopher.beem@maths.ox.ac.uk.`