Problem Sheet 2, General Relativity 2, HT 2018

The starred question is harder and does not contribute to your grades. However, you are encouraged to attempt it. (Students who submit solutions to this question will receive feedback from your TA.)

1. * Assuming Frobenius' theorem, show that, in 4 dimensions, a pair of linearly independent covectors V_a and W_a are orthogonal to a family of two-surfaces iff the two 'twist scalars' $V_{[a}W_b\nabla_c V_{d]}$ and $V_{[a}W_b\nabla_c W_{d]}$ vanish.

[Hint: first show that if the twist scalars vanish, then $\nabla_{[a}V_{b]} = \alpha_{[a}V_{b]} + \beta_{[a}W_{b]}$ for some α_a and β_a, etc., and proceed as for the hypersurface-orthogonal case.]

2.

a) A scalar field φ satisfies the wave equation $\Box\varphi = \nabla^a\nabla_a\varphi = 0$.

Define the tensor field

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\nabla^c\varphi\nabla_c\varphi \ .$$

Show that T_{ab} satisfies the conservation equation $\nabla^a T_{ab} = 0$ as a consequence of the wave equation on φ.

Show that in Minkowski space, with coordinates $x^a = (t, x, y, z)$,

$$T_{00} = \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 + \nabla\varphi\cdot\nabla\varphi\right] \ .$$

b) F_{ab} is an antisymmetric tensor field satisfying the source-free Maxwell equations:

$$\nabla_{[a}F_{bc]} = 0 \ ; \qquad \nabla^a F_{ab} = 0 \ .$$

Define the tensor field

$$T_{ab} = \frac{K}{4}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \ .$$

Show that T_{ab} satisfies the conservation equation by virtue of the Maxwell equations. Show that, in Minkowski space,

$$T_{00} = \frac{K}{8}\left(\mathbf{E}\cdot\mathbf{E} + \mathbf{B}\cdot\mathbf{B}\right) \ .$$

3. Suppose T_{ab} is a symmetric tensor field satisfying the conservation equation and K^a is a Killing vector. Show that $J_a = T_{ab}K^b$ is a conserved vector, *i.e.*, that $\nabla^a J_a = 0$.

Now suppose that T_{ab} is zero outside some world-tube. Show that, in Minkowski space the corresponding 'charge'

$$Q[K] = \int_{t=\text{const}} J_0 \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

is independent of time.

Using the Killing vectors for flat space from Question 6 in Sheet 1, find 10 charges like $Q[K]$ as integrals of T_{ab}. Can you give them names (like 'total energy', 'total angular momentum', etc.)?

4. For a system in flat space with a stress-tensor T_{ab} satisfying the conservation equation define

$$P^i = \int T^{0i}\mathrm{d}^3x\,; \qquad D^i = \int T^{00}x^i\mathrm{d}^3x\,; \qquad Q^{ij} = \int T^{00}x^ix^j\mathrm{d}^3x \ .$$

Show that

$$\frac{dD^i}{dt} = P^i\,; \qquad \frac{dP^i}{dt} = 0\,; \qquad \frac{d^2Q^{ij}}{dt^2} = 2\int T^{ij}\mathrm{d}^3x \ .$$

5. Isothermal Coordinates:

Let M be a two dimensional smooth differentiable manifold with metric g. Isothermal coordinates are coordinates (α, β), in terms of which the metric takes the form

$$\mathrm{d}s^2 = \Omega^2(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2) \ .$$

A solution of the equation $\nabla_a \nabla^a \alpha = 0$ is called a *harmonic function* (in any dimension). In the case where M is two dimensional with positive definite metric, let α be harmonic and let ϵ_{ab} be an antisymmetric tensor field satisfying $\epsilon_{ab}\epsilon^{ab} = 2$. Show that ϵ is covariantly constant.

Consider the equation $\nabla_a \beta = \epsilon_{ab} \nabla^b \alpha$. Show that locally there exists a solution β, and that β is also harmonic. β is called the harmonic function *conjugate* to α.

By choosing α and β as coordinates, show that the metric takes the form $\mathrm{d}s^2 = \Omega^2(\mathrm{d}\alpha^2 + \mathrm{d}\beta^2)$, where Ω is a smooth function.

If x and y are isothermal coordinates on a 2-manifold and $z = x + iy$, show that X and Y are also isothermal coordinates where $X + iY = f(z)$ and f is holomorphic.

A 2-dimensional metric is given by

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sinh^2\theta \, \mathrm{d}\phi^2 \ .$$

Show that x and y are isothermal coordinates, where $x + iy = \tanh(\theta/2)\mathrm{e}^{i\phi}$.

(*All you need to do here is calculate and consider* $\mathrm{d}x^2 + \mathrm{d}y^2$.)

6. Show that if in a curved space-time each coordinate function x^a satisfies the wave equation $\Box\varphi = 0$, then the Christoffel symbols satisfy $g^{ab}\Gamma^c_{ab} = 0$.

Deduce that in these coordinates $\partial_b(g^{ab}g^{1/2}) = 0$, where $g = |\mathrm{det} g_{ab}|$.

(These are often known as *harmonic* coordinates.)

[One way is to use the formula $\Box\,\varphi = g^{-1/2}\partial_a(g^{1/2}g^{ba}\partial_b\phi)$, for the operator $\Box$.]

7. Verify the following identities (for section 2.2 in the lecture notes):

(i) $(T^{lm}x^i x^j)_{,\,lm} = 2T^{ij}$ (which implies $\int T^{ij}\mathrm{d}^3x = 0$).

Show also that $(T^{ij}x^j)_{,\,i} = T^{ii}$ (so $\int T^{ii}\mathrm{d}^3x = 0$).

(ii) $(T^{0i}x^j x^k)_{,\,i} = 2T^{0(j}x^{k)}$ (which implies $\int T^{0(i}x^{j)}\mathrm{d}^3x = 0$).

(iii) $\left(T^k_i x^i x^j - \frac{1}{2}T^{jk}x_i x^i\right)_{,\,k} = T^i_i x^j$ (which gives $\int T^i_i x^j \mathrm{d}^3x = 0$).

(iv)

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right) \qquad \text{where} \qquad r^2 = \mathbf{x}\cdot\mathbf{x} \ .$$