General Relativity II

Problem Set 2

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Fri Wk 5 15:00 - 16:30.

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$\boxed{1}$

Define $U = \text{span}\{V, W\}$ and ~~$\forall$~~ $U^\perp$ the perpendicular space of U such that

$$\tilde{u}^a u_a = 0 \quad \forall\ \tilde{u} \in U^\perp \text{ and } u \in U$$

then consider $X, Y \in U^\perp$

the twist scalar $V_{[a} W_b \nabla_c W_{d]} = 0$

$$\Rightarrow \quad X^c Y^d (V_{[a} W_b \nabla_c W_{d]}) = 0 \quad \forall\ X, Y$$

$$\Rightarrow \quad 0 = X^c Y^d (\cancel{V_{[a}} V_a W_{[b} \nabla_c W_{d]} - V_b W_{[c} \nabla_d W_{a]}$$

$X^c V_c = 0$, $Y^d V_d = 0$ $\because X, Y \in U^\perp$, $V \in U$

$$+ \cancel{V_c} W_{[d} \nabla_a W_{b]} - \cancel{V_d} W_{[a} \nabla_b W_{c]})$$

$$= X^c Y^d (V_a W_b \nabla_{[c} W_{d]} + V_a \cancel{W_c} \nabla_{[d} W_{b]}$$

$X^c W_c = 0$, $\cancel{Y^d} W_d = 0$ $\forall$ ~~$X \in$~~ $X, Y \in U^\perp$ $\because W \in U$

$$+ V_a \cancel{W_d} \nabla_{[b} W_{c]} - V_b W_c \nabla_{[d} W_{a]}$$

$$- V_b \cancel{W_d} \nabla_{[a} W_{c]} - V_b W_a \nabla_{[c} W_{d]})$$

$$= \cancel{\forall}\ \cancel{\text{tota}}\ V_{[a} W_{b]} X^c Y^d \nabla_{[c} W_{d]}$$

$\because V_{[a} W_{b]}$ is in general a non-vanishing tensor, and ~~$X^c Y^d$ tot~~ $X^c Y^d \nabla_{[c} W_{d]}$ is a scalar

$$\therefore \quad X^c Y^d \nabla_{[c} W_{d]} = 0 \quad \forall\ X, Y \in U^\perp$$

Proposition: $X^c Y^d \nabla_{[c} W_{d]} = 0 \quad \forall\ X, Y \in U^\perp$

$$\Leftrightarrow \nabla_{[c} W_{d]} = \gamma_{[c} V_{d]} + \lambda_{[c} W_{d]}$$

where γ_c, λ_c are arbitrary covectors.

"$\Leftarrow$" : $X^c Y^d \nabla_{[c} W_{d]} = X^c Y^d (\gamma_{[c} V_{d]} + \lambda_{[c} W_{d]})$

$$= \frac{1}{2}(\underbrace{X^c\gamma_c Y^d V_d}_{=0} - \underbrace{X^c V_c Y^d \gamma_d}_{=0}$$
$$+ \cancel{2} X^c\lambda_c \underbrace{Y^d W_d}_{=0} - \underbrace{X^c W_c}_{=0} Y^d \lambda_d)$$

$$= 0 \qquad \square$$

"$\Rightarrow$" : Assume $\nabla_{[c} W_{d]} = \gamma_{[c} V_{d]} + \lambda_{[c} W_{d]} + A_{[c} B_{d]}$

such that $A, B \notin U = \mathrm{Span}\{V, W\}$

then $X^a A_a \neq 0$ for some $X \in U^\perp$
$X^a B_a \neq 0$ for some $X \in U^\perp$

$\therefore$ $X^c Y^d A_{[c} B_{d]} = X^c A_c Y^d B_d - X^c B_c Y^d A_d$

$\neq 0$ for some $X, Y \in U^\perp$

$\Rightarrow$ must have ~~X = Y~~ $A, B = 0$

and thus $\nabla_{[c} W_{d]} = \gamma_{[c} V_{d]} + \lambda_{[c} W_{d]}$ $\square$.

By exactly same ~~proc~~ procedure, from

$V_{[a} W_b \nabla_c V_{d]} = 0 \rightarrow W_{[a} V_b \nabla_c V_{d]} = 0$,

we can deduse that

$$x^a y^b \nabla_{[a} V_{b]} = 0 \quad \cancel{and} \quad \forall\ x, y \in U^\perp$$

and that $\nabla_{[a} V_{b]} = \alpha_{[a} V_{b]} + \beta_{[a} W_{b]}$

consider Lie brackets: , for $x, y \in U^\perp$, $V \in U$

$$[X, Y]^a V_a = (X^b \nabla_b Y^a - Y^b \nabla_b X^a) V_a$$

$$= X^b (\nabla_b \underbrace{(Y^a V_a)}_{=0} - Y^a \nabla_b V_a) - Y^b (\nabla_b \underbrace{(X^a V_a)}_{=0} - X^a \nabla_b V_a)$$

$$= -X^a Y^b (\nabla_a V_b - \nabla_b V_a)$$

$$= -2 X^a Y^b \nabla_{[a} V_{b]} = -2 X^a Y^b (\alpha_{[a} V_{b]} + \beta_{[a} W_{b]})$$

$$= 0$$

Similarly $[X, Y]^a W_a = 0$.

$\Rightarrow$ $[X, Y] \in U^\perp$ just like X and Y

Hence by Frobenius theorem, which states that ~~for~~ $W = \mathrm{span}\{u_1, \dots, u_k\}$ and $[u_i, u_j] \in W$ $\forall\ u_i, u_j \in W$ iff W is tangent to a family of k-dimensional surfaces parameterised by $\{u_1, \dots, u_k\}$ and given by $\{w^1, \dots, w^{n-k}\} = \text{const}$.

$\Rightarrow$ in this case $W = \mathrm{span}\{X, Y\}$, $k = 2$, $n = 4$
$= U^\perp$

$\Rightarrow$ $U^\perp$ is tangent to a family of 2-surfaces.

$\therefore \text{span}\{V, W\} = U$

$\therefore$ V, W orthogonal to ~~X~~ all $x \in U^\perp$, which are tangents to a family of 2-surface

$\Rightarrow$ V, W are orthogonal to ~~£~~ a family of 2-surfaces.

$\square$.

$\rightarrow$ The converse is also true since, if V, W $\perp$ to a family of 2-surfaces ~~by Frobenius~~ then the 2-surfaces can be parameterised by X, Y tangent to the surface such that $X, Y \in U^\perp = \text{span}\{V, W\}^\perp$

By Frobenius, ~~X, Y~~ $[X, Y] \in U^\perp$ and

$$[X, Y]^a V_a = 0 \quad , \quad [X, Y]^a W_a = 0$$

$$\Rightarrow \quad X^a Y^b \nabla_{[a} V_{b]} = 0 \, , \, X^a Y_b \nabla_{[a} W_{b]} = 0$$

$$\Rightarrow \quad \nabla_{[a} V_{b]} = \alpha_{[a} V_{b]} + \beta_{[a} W_{b]}$$

$\therefore$ ~~$V_a W_b \nabla_{[c} W_{d]}$~~

$$V_a W_b \nabla_{[c} V_{d]} = V_a W_b \alpha_{[c} V_{d]} + V_a W_b \beta_{[c} W_{d]}$$

$$\Rightarrow \quad V_{[a} W_b \nabla_c V_{d]} = V_{[a} W_b \alpha_c V_{d]} + V_{[a} W_b \beta_c W_{d]}$$

$\because V_a V_d$ is symmetric

and $W_b W_d$ is also symmetric

$\therefore$ Antisymmetrisation gives 0

$\Rightarrow$ $\cancel{V_a W_b \nabla_c W_d}$ $V_{[a} W_d \nabla_c V_{d]} = 0$

Similarly $V_{[a} W_b \nabla_c W_{d]} = 0$

$\Rightarrow$ twist scalars vanish $\square$

$\boxed{2}$ a)

$$\Box\varphi = \nabla^a\nabla_a\varphi = 0$$

$$T_{ab} = \nabla_a\varphi\nabla_b\varphi - \frac{1}{2}g_{ab}\nabla^c\varphi\nabla_c\varphi$$

$$\nabla^a T_{ab} = \nabla^a(\nabla_a\varphi\nabla_b\varphi) - \frac{1}{2}g_{ab}\nabla^a(\nabla^c\varphi\nabla_c\varphi)$$

$$= \cancel{\nabla^a\varphi}\,\cancel{(\underbrace{\nabla^a\nabla_a\varphi}_{=0})\nabla_b\varphi} + (\nabla_a\varphi)(\nabla^a\nabla_b\varphi)$$

$$- \frac{1}{2}g_{ab}\left((\nabla^a\nabla^c\varphi)(\nabla_c\varphi) + \nabla^c\varphi(\nabla^a\nabla_c\varphi)\right)$$

$$= 2(\nabla^c\varphi)(\nabla^a\nabla_c\varphi)$$

$$= \cancel{(\nabla_c\varphi)(\nabla^c\nabla_b\varphi)}$$

$$= \cancel{\nabla\nabla_c\varphi}$$

$$= (\nabla^c\varphi)(\nabla_c\nabla_b\varphi) - \frac{1}{2}\times 2\times g_{ab}(\nabla^c\varphi)(\nabla^a\nabla_c\varphi)$$

$$= \nabla^c\varphi(\underbrace{\nabla_c\nabla_b - \nabla_b\nabla_c}_{=0})\varphi \quad \because \varphi \text{ is scalar}$$

$$= \underline{\underline{0}}$$

$$\Rightarrow \quad \nabla^a T_{ab} = 0$$

□

in Minkowski space $\nabla_a = \partial_a$, $(\Gamma^a_{bc} = 0)$

$g_{ab} = \eta_{ab} = (-1, +1, +1, +1)$

$$\therefore\ T_{00} = \partial_0\varphi\,\partial_0\varphi - \tfrac{1}{2}\eta_{00}\,\partial^c\varphi\,\partial_c\varphi$$

$$= \left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2}\left(-\left(\frac{\partial\varphi}{\partial t}\right)^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi\right)$$

$$= \frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial t}\right)^2 + \vec{\nabla}\varphi\cdot\vec{\nabla}\varphi\right]$$

$\square$

b) $\nabla_{[a}F_{bc]} = 0 \qquad \nabla^a F_{ab} = 0.$

$$T_{ab} = \frac{K}{4}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right)$$

$$\Rightarrow \nabla^a T_{ab} = \frac{K}{4}\left(\nabla^a(F_{ac}F_b{}^c) - \frac{1}{4}g_{ab}\nabla^a(F_{cd}F^{cd})\right)$$

$$= \frac{K}{4}\Big(\underbrace{\cancel{(\nabla^a F_{ac})}}_{=0}F_b{}^c + F_{ac}(\nabla^a F_b{}^c) - \frac{1}{4}g_{ab}(\nabla^a F_{cd})F^{cd}$$
$$- \frac{1}{4}g_{ab}F_{cd}(\nabla^a F^{cd})\Big)$$

$$= \frac{K}{4}\left[F^{ac}(\nabla_a F_{bc}) - \frac{1}{2}F^{cd}(\nabla_b F_{cd})\right]$$

$\nabla_{[a}F_{bc]} = 0$ }

you could use that here & you would be done.

$$= \frac{K}{4}\left(F^{ac}(\nabla_a F_{bc}) + \frac{1}{2}F^{cd}(\nabla_c F_{db} + \nabla_d F_{bc})\right)$$

$$\cancel{= \frac{K}{4}\left(F^{ac}(\nabla_a F_{bc}) + \frac{1}{2}F^{ac}(\nabla_c F_{ab} + \nabla_a F_{bc}\right.}$$

$F_{cd} = -F_{dc}$ }

$$= \frac{K}{4}\left(F^{ac}(\nabla_a F_{bc}) + \frac{1}{2}F^{cd}(\nabla_c F_{db}) + \frac{1}{2}F^{dc}(\nabla_d F_{cb})\right)$$

$$= \frac{K}{4}\left(F^{ac}(\nabla_a F_{bc}) + F^{cd}(\nabla_c F_{db})\right)$$

$$= \frac{K}{4}\left(F^{ac}(\cancel{\nabla_a F_{bc}}) - F^{cd}(\cancel{\nabla_c F_{bd}})\right)$$

$$= \underline{\underline{0}} \quad \square$$

$$T_{00} = \frac{K}{4}\left(F_{0c}F_0{}^c - \frac{1}{4} g_{00} F_{cd}F^{cd}\right)$$

in Minkowski spacetime $\eta_{ab} = g_{ab} = (-1,1,1,1)$

$$T_{00} = \frac{K}{4}\left(\eta^{ab} F_{0a}F_{0b} + \frac{1}{4} F_{cd}F^{cd}\right)$$

$$F_{ab} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\therefore F_{cd}F^{cd} = 2(B^2 - E^2) = 2(\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E})$$

$$\eta^{ab} F_{0a}F_{0b} = \vec{E}\cdot\vec{E}$$

$$\therefore T_{00} = \frac{K}{4}\left(\vec{E}\cdot\vec{E} + \frac{1}{2}(\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E})\right)$$

$$= \frac{K}{4}\left(\frac{1}{2}\vec{E}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{B}\right)$$

$$= \frac{K}{8}\left(\vec{E}\cdot\vec{E} + \vec{B}\cdot\vec{B}\right)$$

$\square$

$\boxed{3}$. $T_{ab} = T_{ba}$ $\quad \nabla^a T_{ab} = 0$, K^a Killing vector

$\Rightarrow \nabla_{(a} K_{b)} = 0$

i.e. $J_a = T_{ab} K^b$

$$\therefore \nabla^a J_a = \nabla^a (T_{ab} K^b) = \underbrace{(\nabla^a T_{ab})}_{=0} K^b + T_{ab} \nabla^a K^b$$

$$= T_{ab} \underbrace{\nabla^{(a} K^{b)}}_{=0} + T_{ab} \underbrace{\nabla^{[a} K^{b]}}_{=0}$$

$=0$ ∵ T_{ab} symmetric

$\nabla^{[a}K^{b]}$ antisymmetric

$= 0$ //

$\blacksquare$

In Minkowski space $\quad \nabla^a J_a = 0 \Rightarrow \frac{\partial J_0}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$.

$$\therefore \frac{\partial \mathcal{Q}}{\partial t} = \int \frac{\partial J_0}{\partial t} d^3x = -\int \vec{\nabla} \cdot \vec{J} \, d^3 x.$$

$$= -\int_{\partial V} \underbrace{\vec{J} \cdot d\vec{S}}_{} = 0 \quad //$$

What are we assuming on the boundary?

State this

$J_i = T_{ib} K^b = 0$

∵ $T_{ab} = 0$ ~~out~~ outside

some world tube

$\therefore \frac{\partial \mathcal{Q}}{\partial t} = 0 \Rightarrow \mathcal{Q}$ independent of time.

Killing vectors from $\boxed{6}$ of PS 1:

$P_{(0)} = (1, 0, 0, 0)$: $\quad J_a = T_{ab} P_{(0)}^b$

$= T_{a0} \times (1) = T_{a0}$

$\therefore J_0 = T_{00}$ $\qquad Q = \int T_{00}\, d^3x.$

$= \text{total energy.} = E$

$P_{(1)} = (0,1,0,0)$ $\qquad J_a = T_{ab} P_{(1)}^b = T_{a1} \times (1) = T_{a1}$

$J_0 = T_{01}$

$$Q = \int T_{01}\, d^3x = P_x$$

$= \text{momentum in "x" direction.}$

Similarly

$P_{(2)} = (0,0,1,0) \rightarrow Q = \int T_{02}\, d^3x = P_y$ y-momentum

$P_{(3)} = (0,0,0,1) \rightarrow Q = \int T_{03}\, d^3x = P_z$ z-momentum

$J_{(1)} = (0,0,\mp z,\pm y)$: $\qquad J_a = T_{ab} J_{(1)}^b = +T_{a2} \underbrace{z}_{x_3} \mp T_{a3} \underbrace{y}_{x_2}.$

What's the Killing vector here? generally?

$J_0 = +T_{02} x_3 \mp T_{03} x_2$

$$Q = \int -T_{02} x_3 + T_{03} x_2\, d^3x = x\text{-angular momentum} = L_x$$

Similarly $J_{(2)} = (0, \mp z, 0, \pm x) \rightarrow Q = \int -T_{01} x_3 + T_{03} x_1\, d^3x$

$= y\text{-angular momentum} = L_y$

$J_{(3)} = (0, +y, \mp x, 0) \rightarrow Q = \int T_{01} x_2 - T_{02} x_1\, d^3x$

$= z\text{-angular momentum} = L_z$

Boosts:

$$K_{(1)} = (-x, +t, 0, 0) \qquad \cancel{\text{Fa}}\ J_a = T_{ab} K_{(1)}^b$$

$$= -T_{a0} X + T_{a1} t.$$

$$\therefore J_0 = -T_{00} X + T_{01} t$$

$$\therefore Q = \int -T_{00} X + T_{01} t \, d^3x.$$

$$= \int -T_{00} x \, d^3x + t \underbrace{\int T_{01} \, d^3x}_{\cancel{P_{(1)}}\ P_x}$$

$$= \int -T_{00} X \, d^3x + t P_x$$

$$\Rightarrow \frac{Q}{E} = - \underbrace{\frac{\int T_{00} X \, d^3x}{\int T_{00} \, d^3x}} + \frac{t P_x}{E} \equiv -X_{cm}(0)$$

Centre of mass position $X_{cm}(t)$ in X-coordinate

$$\therefore X_{cm}(t) = X_{cm}(0) + \underbrace{\frac{P_x}{E}}_{\text{cm velocity in } X} t$$

$\therefore$ Q represents the CM position at $t=0$ of X.

Similarly

$$K_{(2)} = (\cancel{x} -Y, 0, t, 0) \rightarrow Q = \int -T_{00} Y + T_{02} t \, d^3x$$

$\rightarrow$ CM position of Y at $t=0$

$K_{(3)} = (-z, 0, 0, t) \rightarrow Q = \int -T_{00} z + T_{03}\, t \, d^3x.$

$\rightarrow$ CM position of z at $t=0$

$\boxed{4}$

$$p^i = \int T^{0i} d^3x \qquad D^i = \int T^{00} x^i d^3x \qquad Q^{ij} = \int T^{00} x^i x^j d^3x.$$

$$\rightarrow \frac{dD^i}{dt} = \int \underbrace{\frac{dT^{00}}{dt}}_{\frac{dT^{00}}{dt} = -\partial_j T^{0j}} x^i d^3x = -\int \partial_j T^{0j} x^i d^3x.$$

$$= -\underbrace{\int \partial_j (T^{0j} x^i) d^3x}_{=0 \text{ assuming boundary terms vanish}} + \int T^{0j} \underbrace{\partial_j x^i}_{\delta_j{}^i} d^3x$$

$$= \int T^{0i} d^3x = P^i \quad \square$$

$$\rightarrow \frac{dP^i}{dt} = \int \underbrace{\frac{dT^{0i}}{dt}}_{\frac{dT^{0i}}{dt} = -\partial_j T^{ji}} d^3x = -\int \underbrace{(\partial_j T^{ji}) d^3x}_{\text{assuming boundary terms vanish.}} = 0$$

$\square$

$$Q^{ij} = \int T^{00} x^i x^j \, d^3x .$$

$$\frac{dQ^{ij}}{dt} = \int \frac{dT^{00}}{dt} x^i x^j \, d^3x = -\int (\partial_k T^{0k}) x^i x^j \, d^3x .$$

$$= -\int \underbrace{\partial_k (T^{0k} x^i x^j)}_{\Rightarrow 0} d^3x + \int T^{0k} \partial_k (x^i x^j) \, d^3x .$$

$$= \int T^{0k} (\delta_k{}^i x^j + \delta_k{}^j x^i) \, d^3x .$$

$$= \int T^{0i} x^j + T^{0j} x^i \, d^3x .$$

$$\rightarrow \frac{d^2 Q^{ij}}{dt^2} = \int \frac{dT^{0i}}{dt} x^j + \frac{dT^{0j}}{dt} x^i \, d^3x .$$

$$= -\int (\partial_k T^{ki}) x^j d^3x - \int (\partial_k T^{kj}) x^i d^3x$$

$$= -\int \partial_k (\underbrace{T^{ki} x^j}_{\to 0}) \, d^3x - \int \partial_k (\underbrace{T^{kj} x^i}_{\to 0}) \, d^3x .$$

$$+ \int T^{ki} \underbrace{\partial_k x^j}_{\delta_k{}^j} d^3x + \int T^{kj} \underbrace{\partial_k x^i}_{\delta_k{}^i} d^3x$$

$$= \int T^{ji} + T^{ij} \, d^3x \underbrace{=}_{T^{ij} = T^{ji}} 2 \int T^{ij} d^3x$$

$\boxed{5}$ $\because \varepsilon_{ab}$ is completely antisymmetric

$\therefore \varepsilon_{ab} = e\tilde{\varepsilon}_{ab}$ where $\tilde{\varepsilon}_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab}$

$$2 = \varepsilon_{ab}\varepsilon^{ab} = e^2 \tilde{\varepsilon}_{ab}\tilde{\varepsilon}^{ab} = e^2 \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{cd}\, g^{ac} g^{bd}$$

$$= 2e^2 \left(\frac{1}{2!} \tilde{\varepsilon}_{ab}\tilde{\varepsilon}_{cd} (g^{-1})^{ac} (g^{-1})^{bd} \right) \quad \Big|\ g^{-1} \because \text{indices } \cancel{\text{}} \text{ are raised}$$

the ( ) is precisely the definition of $\det(g^{-1})$

$$\therefore \quad 1 = e^2 \det(g^{-1}) = \frac{e^2}{\det(g)} \qquad \therefore e^2 = \det(g)$$

choose convention $\varepsilon_{12} > 0$ $\Rightarrow$ $\varepsilon_{ab} = \sqrt{\det(g)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab}$

(Problem states that ε_{ab} is a tensor, but ~~to~~ it can be shown. For proof ~~the~~ see Question 4 of GR I problem sheet 1)

$\Rightarrow$ $\nabla_c \varepsilon_{ab}$ is a $(0,3)$ tensor

Evaluate it in normal coordinates, i.e. choose ~~to~~ local coordinates such that the metric is ~~Riemannian~~ Riemannian (Not Lorentzian because g is positive definite), $\rightarrow$ $g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ab}$ and

$\Gamma^a{}_{bc} = 0$, $\nabla_c = \partial_c$, $\det(g) = 1$

The in this coordinate $\nabla_c \varepsilon_{ab} = \partial_c \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{ab} = 0$

$\because$ Both sides of above equation are tensors

$\therefore$ Above equation true in any coordinate systems (frames)

$\therefore \nabla_c \varepsilon_{ab} = 0$ ✓ $\Rightarrow$ covariantly constant.

~~OR $\because$ 2 is a scalar number~~

$\cancel{\therefore \nabla_e(2) = 0 = \nabla_e(\varepsilon_{ab}\varepsilon^{ab}) = \nabla_e(\varepsilon_{ab}\varepsilon_{cd}\, g^{ac} g^{bd})}$

$\cancel{= g^{ac} g^{bd} (\varepsilon_{ab} \nabla_e \varepsilon_{cd} + \varepsilon_{cd} \nabla_e \varepsilon_{ab})}$

$\rightarrow \quad \nabla_a \beta = \varepsilon_{ab} \nabla^b \alpha$

Locally metric & $g_{ab} = \delta_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \varepsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\nabla^b = \partial^b = \partial_b \quad , \quad \nabla_a = \partial_a$

$$\therefore \quad \partial_a \beta = \varepsilon_{ab} \partial_b \alpha \Rightarrow \begin{cases} \partial_1 \beta = \partial_2 \alpha \\ \partial_2 \beta = -\partial_1 \alpha \end{cases}$$

$$\Rightarrow \quad \partial_1^2 \beta + \partial_2^2 \beta = (\partial_1 \partial_2 - \partial_2 \partial_1)\alpha = 0$$

$\Rightarrow \quad \nabla^2 \beta = 0 \Rightarrow$ locally β satisfies the

Laplace's equation $\Rightarrow$ there must be a solution

$$\nabla^a \nabla_a \beta = \nabla^a (\varepsilon_{ab} \nabla^b \alpha) \underbrace{=}_{\nabla_c \varepsilon_{ab} = 0} \varepsilon_{ab} \nabla^a \nabla^b \alpha$$

$\because$ α is a ~~scalar~~ function with No index

$\therefore \quad \nabla^a \nabla^b \alpha = \nabla^b \nabla^a \alpha \Rightarrow \nabla^a \nabla^b \alpha$ symmetric.

Also $\because$ ε_{ab} is an antisymmetric tensor

$$\therefore \quad \varepsilon_{ab} \nabla^a \nabla^b \alpha = -\varepsilon_{ba} \nabla^b \nabla^a \alpha = 0$$

$\Rightarrow \quad \nabla^a \nabla_a \beta = 0 \Rightarrow \beta$ is also harmonic ✓

If ~~$a=$~~ $x^1=\alpha$, $x^2=\beta$ are coordinates and choose the convention $\varepsilon_{ab}=\begin{pmatrix} 0 & -\sqrt{\det(g)} \\ \sqrt{\det(g)} & 0 \end{pmatrix}$.

$$\nabla_1\beta=\partial_1\beta=\nabla_2\alpha=\partial_2\alpha=\frac{\partial\beta}{\partial\alpha}=\frac{\partial\alpha}{\partial\beta}=0.$$

$$\nabla_1\alpha=\partial_1\alpha=\frac{\partial\alpha}{\partial\alpha}=\nabla_2\beta=\partial_2\beta=\frac{\partial\beta}{\partial\beta}=1$$

equation $\nabla_a\beta=\varepsilon_{ab}\nabla^b\alpha=\varepsilon_{ab}g^{bc}\nabla_c\alpha$

→ when $a=1$, $0=\nabla_1\beta=\underbrace{\varepsilon_{11}}_{=0}\cancel{g^{1c}\nabla_c\alpha}+\varepsilon_{12}g^{2c}\nabla_c\alpha$

$$=\varepsilon_{12}g^{21}\nabla_1\alpha+\cancel{\varepsilon_{12}g^{22}\underbrace{\nabla_2\alpha}_{=0}}$$

$$=\varepsilon_{12}g^{21}\underbrace{\nabla_1\alpha}_{1}=-\sqrt{\det(g)}\,g^{21}$$

$\Rightarrow g^{21}=0$ $\because$ g^{ab} is symmetric $\therefore\ \underline{\underline{g^{12}=g^{21}=0}}$

$\therefore\ g_{ab}=\begin{pmatrix} A & 0 \\ 0 & C\end{pmatrix}$ $g^{ab}=\begin{pmatrix} \frac{1}{A} & 0 \\ 0 & \frac{1}{C}\end{pmatrix}$

→ when ~~$a=2$~~ $a=2$,

$$1=\nabla_2\beta=\cancel{\varepsilon_{22}\nabla^2\alpha}+\varepsilon_{21}g^{11}\underbrace{\nabla_1\alpha}_{=1}+\cancel{\varepsilon_{21}\underbrace{g^{12}}_{=0}\underbrace{\nabla_2\alpha}_{=0}}$$

$$=\varepsilon_{21}g^{11}$$

$$\varepsilon_{ab}=\begin{pmatrix} 0 & -\sqrt{\det(g)} \\ \sqrt{\det(g)} & 0\end{pmatrix}=\begin{pmatrix} 0 & -\sqrt{AC} \\ \sqrt{AC} & 0\end{pmatrix}$$

$$\therefore\ 1=\varepsilon_{21}g^{11}=(\sqrt{AC})\left(\frac{1}{A}\right)=\sqrt{\frac{C}{A}}$$

$\therefore \frac{C}{A} = 1 \Rightarrow A = C$ ✓

$$g_{ab} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \text{For } g \text{ to be positive definite, require } A > 0$$

$\therefore$ let $A = \Omega^2$, we have

$$g_{ab} = \begin{pmatrix} \Omega^2 & 0 \\ 0 & \Omega^2 \end{pmatrix} \quad \text{and} \quad ✓$$

$$\underline{\underline{ds^2 = \Omega^2 (d\alpha^2 + d\beta^2)}}$$

→ If $f(z) = X + iY$ where $z = x + iy$ is holomorphic, then we have Cauchy-Riemann equations

$$\frac{\partial X}{\partial x} = \frac{\partial Y}{\partial y}, \quad \frac{\partial X}{\partial y} = -\frac{\partial Y}{\partial x}$$

$$\therefore d^2X + d^2Y = \left(\frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy\right)^2 + \left(\frac{\partial Y}{\partial x}dx + \frac{\partial Y}{\partial y}dy\right)^2$$

$$= \left(\left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2\right)dx^2$$

$$+ \left(\left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2\right)dy^2$$

$$+ 2\left(\frac{\partial X}{\partial x}\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y}\right)dx\,dy$$

Use Cauchy-Riemann equations $\Rightarrow$

$$\rightarrow \left(\frac{\partial X}{\partial x}\right)^2 + \left(\frac{\partial Y}{\partial x}\right)^2 = \left(\frac{\partial Y}{\partial y}\right)^2 + \left(-\frac{\partial X}{\partial y}\right)^2$$

$$= \left(\frac{\partial X}{\partial y}\right)^2 + \left(\frac{\partial Y}{\partial y}\right)^2 \equiv A^2 \geq 0$$

$$\Rightarrow \frac{\partial X}{\partial x}\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x}\frac{\partial Y}{\partial y} = \frac{\partial X}{\partial x}\frac{\partial X}{\partial y} + \left(-\frac{\partial X}{\partial y}\right)\left(\frac{\partial X}{\partial x}\right)$$

$$= 0$$

$$\therefore \; d^2X + d^2Y = A^2(d^2x + d^2y)$$

$\because$ x, y are isothermal coordinates

$$\therefore \; d^2s = \Omega^2(d^2x + d^2y) = \left(\frac{\Omega}{A}\right)^2(d^2X + d^2Y)$$

$\Rightarrow$ X, Y are also isothermal coordinates.

$\Rightarrow$ for $x + iy = \tanh(\theta/2)e^{i\phi} =$

$$= \underbrace{\tanh(\theta/2)\cos\phi}_{x} + i(\underbrace{\tanh(\theta/2)\sin\phi}_{y})$$

Consider:

$$d^2x + d^2y = \left(\frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \phi}d\phi\right)^2 + \left(\frac{\partial y}{\partial \theta}d\theta + \frac{\partial y}{\partial \phi}d\phi\right)^2$$

$$+\left(\frac{1}{2}\operatorname{sech}^2\left(\tfrac{\theta}{2}\right)\sin\phi \, d\theta + \tanh\left(\tfrac{\theta}{2}\right)\cos\phi \, d\phi\right)^2.$$

$$= \left(\frac{1}{4}\operatorname{sech}^4\left(\tfrac{\theta}{2}\right)\underbrace{(\cos^2\phi + \sin^2\phi)}_{=1}\right) d\theta^2$$

$$+ \left(\tanh^2\left(\tfrac{\theta}{2}\right)\underbrace{(\sin^2\phi + \cos^2\phi)}_{=1}\right) d\phi^2$$

$$+ \left(\frac{1}{2}\operatorname{sech}^2\left(\tfrac{\theta}{2}\right)\tanh\left(\tfrac{\theta}{2}\right)\right)\underbrace{\left(\cancel{-2\cos\phi\sin\phi} \cancel{+ 2\sin\phi\cos\phi}\right)}_{=0} d\theta \, d\phi$$

$$= \frac{1}{4}\cdot\frac{1}{\cosh^4(\theta/2)} d\theta^2 + \tanh^2(\theta/2)\, d\phi^2$$

$$= \frac{1}{4}\left(\frac{1}{\cosh^4(\theta/2)}\right)\left(d\theta^2 + \left(2\sinh(\theta/2)\cosh(\theta/2)\right)^2 d\phi^2\right)$$

$$= \frac{1}{4\cosh^2(\theta/2)}\left(d\theta^2 + \sinh^2(\theta)\, d\phi^2\right)$$

$$= \frac{1}{4\cosh^2(\theta/2)} ds^2 \qquad \checkmark$$

$$\Rightarrow ds^2 = 4\cosh^2(\theta/2)\,(dx^2 + dy^2)$$

$$= F(x,y)\,(dx^2 + dy^2)$$

$F = 4\cosh^2\left(\frac{\theta}{2}\right)$ = some function of x and y

$\Rightarrow$ x, y are isothermal coordinates. ∎

$\boxed{6}$ $\quad \Box = \nabla^a \nabla_a$

we have $\nabla^a \nabla_a X^\mu = 0$ for each μ, Note that we do not treat X^μ as a vector, we treat each component of X^μ as a function with no index. ($\neq$ single component).

$$\therefore \nabla_a X^\mu = \partial_a X^\mu = \frac{\partial X^\mu}{\partial x^a} = \delta^\mu_a$$

$$0 = \Box X^\mu = \nabla^a \nabla_a X^\mu = g^{ab} \nabla_b \nabla_a X^\mu = g^{ab} \nabla_b \delta^\mu_a$$

$$= g^{ab} \left(\underbrace{\partial_b \delta^\mu_a}_{=0} - \Gamma^c_{ab} \delta^\mu_c \right)$$

$$= g^{ab} \left(- \Gamma^c_{ab} \delta^\mu_c \right) = - g^{ab} \Gamma^\mu_{ab}$$

$$\Rightarrow \quad \underline{g^{ab} \Gamma^\mu_{ab} = 0} \quad \Box$$

$\because \Box \varphi = g^{-1/2} \partial_a \left(g^{1/2} g^{ba} \partial_b \varphi \right)$, $g = |\det(g)|$.

For $\varphi = X^\mu$

$$0 = \Box X^\mu = g^{-\frac{1}{2}} \partial_a \left(g^{1/2} g^{ba} \underbrace{\partial_b X^\mu}_{\delta^\mu_b} \right)$$

$$= g^{-\frac{1}{2}} \partial_a \left(g^{1/2} g^{ba} \delta^\mu_b \right)$$

$$= g^{-\frac{1}{2}} \partial_a \left(g^{1/2} g^{a\mu} \right) \Rightarrow \partial_a \left(g^{1/2} g^{a\mu} \right) = 0$$

$$\Rightarrow \quad \underline{\partial_b \left(g^{b\mu} g^{1/2} \right) = 0} \quad \Box$$

*(section 2.2 of lecture notes refers to static

$\boxed{7}$ ~~case~~ case where $\partial_0 = 0$ and $\partial_i T^{ij} = 0$).

(i) $(T^{lm} x^i x^j)_{,lm} = \partial_l \partial_m (T^{lm} x^i x^j)$

$= \partial_l (\underbrace{(\partial_m T^{ml})}_{=0} x^i x^j + T^{lm} \delta^i_m x^j + T^{lm} x^i \delta^j_m)$

$= \partial_l (T^{li} x^j + T^{lj} x^i)$

$= \underbrace{(\partial_l T^{li})}_{=0} x^j + T^{li} \delta^j_l + \underbrace{(\partial_l T^{lj})}_{=0} x^i$

$+ T^{lj} \delta^i_l$

$= T^{ij} + T^{ji} = 2T^{ij}$

Also. $(T^{ij} x^j)_{,i} = \underbrace{(\partial_i T^{ij})}_{=0} x^j + T^{ij} \underbrace{\partial_i x^j}_{\delta_i^j}$

$= T^{ij} \delta_i^j = T^{jj} = T^{ii}$

(ii) $(T^{0i} x^j x^k)_{,i} = \underbrace{\partial_i (T^{0i})}_{=0} x^j x^k$

$+ T^{0i} (\partial_i x^j) x^k + T^{0i} x^j \partial_i x^k.$

$= T^{0i} \delta_i^j x^k + T^{0i} \delta_i^k x^j$

$= T^{0j} x^k + T^{0k} x^j = 2 T^{0(j} x^{k)}$

(iii) $(T^k{}_i x^i x^j - \frac{1}{2} T^{jk} x_i x^i)_{,k} =$

$$= (\overbrace{\partial_k T^k{}_i}^{0}) x^i x^j + T^k{}_i (\overbrace{\partial_k x^i}^{\delta^i_k}) x^j + T^k{}_i x^i (\overbrace{\partial_k x^j}^{\delta_k^j})$$

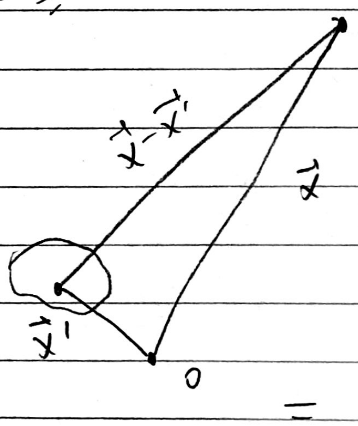
$$- \tfrac{1}{2}(\underbrace{\partial_k T^{jk}}_{0}) x_i x^i - \tfrac{1}{2} T^{jk} (\underbrace{\partial_k x_i}_{\delta_{ki}}) x^i - \tfrac{1}{2} T^{jk} x_i (\underbrace{\partial_k x^i}_{\delta_k^i})$$

$$= T^i{}_i x^j + T^j{}_i x^i - \tfrac{1}{2} T^j{}_i x^i - \tfrac{1}{2} T^{ji} x_i$$

$$= T^i{}_i x^j + (T^j{}_i \underbrace{x^i - \tfrac{1}{2} T^j{}_i x^i - \tfrac{1}{2}}_{=0} T^j{}_i x^i)$$

$$= T^i{}_i x^j \quad \square$$

(N) $|\vec{x}'| \ll |\vec{x}|$



$$\frac{1}{|\vec{x}-\vec{x}'|} = |\vec{x}-\vec{x}'|^{-1}$$

$$= \left((\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')\right)^{-\frac{1}{2}}$$

$$= (\underbrace{\vec{x}\cdot\vec{x}}_{r^2} - 2\vec{x}\cdot\vec{x}' + \vec{x}'\cdot\vec{x}')^{-\frac{1}{2}}$$

$$= \left(r^2\left(1 - 2\frac{\vec{x}\cdot\vec{x}'}{r^2} + \frac{\vec{x}'\cdot\vec{x}'}{r^2}\right)\right)^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left(1 - \frac{2\vec{x}\cdot\vec{x}'}{r^2} + \frac{\vec{x}'\cdot\vec{x}'}{r^2}\right)^{-\frac{1}{2}}$$

$\because |\vec{x}'\cdot\vec{x}'| \ll |\vec{x}\cdot\vec{x}'|$ and $(1+x)^{\frac{1}{2}} \simeq 1 + \frac{1}{2}x + O(x^2)$

$\therefore \frac{1}{|\vec{x}-\vec{x}'|} \simeq \frac{1}{r}\left(1 + \frac{1}{2}\left(\frac{2\vec{x}\cdot\vec{x}'}{r^2}\right) + O\left(\frac{1}{r^2}\right)\right)$

good

$$= \frac{1}{r} + \frac{\vec{x} \cdot \vec{x}'}{r^3} + O\left(\frac{1}{r^3}\right)$$

$\nabla^2 \phi = 0$