General Relativity II

Problem Set 1

Ziyan Li

TA: Mr. Michael Coughlan

Fri Wk3 13:00 - 14:30 C4.

Q	grade
1	A
2	B+
3	A
4	A-
5	A
6	A
7	A

Note:

In $\boxed{6}$, $(A)_e X^e + (B)^f{}_e \underbrace{\nabla_f X^e}_{\substack{\text{antisymmetric} \\ \text{otherwise arbitrary}}}$

$\uparrow$ arbitrary

should imply $A = 0$

and B is symmetric, not $B = 0$

But after taking the trace of B, the same result for R_{abcd} follows as if we set $B = 0$.

~~I get~~ I have no time to correct this, sorry ...

(ref: Weinberg pg. 382)

$\boxed{1}$

$\nabla_{[a}R_{bc]de}$, $R_{bcde} = -R_{cbde}$

$\Rightarrow \quad \nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0$

$\times g^{ce} \quad \Rightarrow \quad \nabla_a R_{bd} - \nabla_b R_{ad} + \nabla_c R_{abd}{}^{c} = 0$

$\Rightarrow \quad \nabla_a R_{bd} - \nabla_b R_{ad} - \nabla^c R_{cdab} = 0$

$R_{ad} = R_{da} \quad \Rightarrow \quad \nabla_a R_{bd} - \nabla_b R_{da} - \nabla^c R_{cdab} = 0$

$\times g^{db} \quad \Rightarrow \quad \nabla_a R - \nabla^b R_{ba} - \nabla^c R_{ca} = 0$

$\Rightarrow \quad \nabla_a R = 2\nabla^c R_{ca}$

$\times \quad \Rightarrow \quad \nabla^b R g_{ba} = 2\nabla^b R_{ba}$

$a \leftrightarrow b \quad \Rightarrow \quad \nabla^a (R g_{ab}) = 2\nabla^a R_{ab}.$

$\times \frac{1}{2} \quad \Rightarrow \quad \nabla^a \left(R_{ab} - \frac{1}{2} R g_{ab} \right) = 0$ □

$\boxed{2}$ $[X,Y]^a = X^b \partial_b Y^a - Y^b \partial_b X^a$

$\therefore \quad [X,[Y,Z]]^a + [Y,[Z,X]]^a + [Z,[X,Y]]^a$

$= \cancel{[X^a, Y^b \partial_b Z^a - Z^b \partial_b Y^a] + [Y^a}$,

Consider using a smooth function f to write $[X,Y](f) = X(Y(f)) - Y(X(f))$

$$\begin{aligned} = \; & X^b \partial_b [Y,Z]^a - [Y,Z]^b \partial_b X^a \\ & + Y^b \partial_b [Z,X]^a - [Z,X]^b \partial_b Y^a \\ & + Z^b \partial_b [X,Y]^a - [X,Y]^b \partial_b Z^a \end{aligned}$$

$$\begin{aligned} = \; & X^b \partial_b (Y^c \partial_c Z^a - Z^c \partial_c Y^a) - (Y^c \partial_c Z^b - Z^c \partial_c Y^b) \partial_b X^a \\ & + Y^b \partial_b (Z^c \partial_c X^a - X^c \partial_c Z^a) - (Z^c \partial_c X^b - X^c \partial_c Z^b) \partial_b Y^a \\ & + Z^b \partial_b (X^c \partial_c Y^a - Y^c \partial_c X^a) - (X^c \partial_c Y^b - Y^c \partial_c X^b) \partial_b Z^a \end{aligned}$$

$= 0$ □

$\boxed{3}$ torsion free $\Rightarrow$ $T^c{}_{ab} = 2\Gamma^c{}_{[ab]} = 0$

$\Rightarrow$ $\Gamma^c{}_{ab} = \Gamma^c{}_{ba}$ $(*)$

$$RHS = X^b \nabla_b Y^a - Y^b \nabla_b X^a$$

$$= X^b(\partial_b Y^a + \Gamma^a{}_{bc} Y^c) - Y^b(\partial_b X^a + \Gamma^a{}_{bc} X^c)$$

$$= X^b \partial_b Y^a - Y^b \partial_b X^a + (\Gamma^a{}_{bc} X^b Y^c - \Gamma^a{}_{bc} X^c Y^b)$$

$$= X^b \partial_b Y^a - Y^b \partial_b X^a + \underbrace{(\Gamma^a{}_{bc} - \Gamma^a{}_{cb})}_{0} X^b Y^c \quad \text{(torsion free)}$$

$$= X^b \partial_b Y^a - Y^b \partial_b X^a$$

$$= [X, Y]^a \quad \square$$

$$(\nabla_{[X,Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X) Z^a$$

$$= ([X,Y]^b \nabla_b - X^c \nabla_c Y^b \nabla_b + Y^c \nabla_c X^b \nabla_b) Z^a$$

$$= \cancel{(X^c \nabla_c Y^b \nabla_b}$$

$$= [X,Y]^b \nabla_b Z^a - X^c \nabla_c Y^b \nabla_b Z^a + Y^c \nabla_c X^b \nabla_b Z^a$$

$$= (X^c \nabla_c Y^b)(\nabla_b Z^a) - (Y^c \nabla_c X^b)(\nabla_b Z^a)$$

$$- X^c \nabla_c (Y^b \nabla_b Z^a) + Y^c \nabla_c (X^b \nabla_b Z^a)$$

$$= \cancel{(X^c \nabla_c Y^b)(\nabla_b Z^a)} - \cancel{(Y^c \nabla_c X^b)(\nabla_b Z^a)}$$

$$- \cancel{(X^c \nabla_c Y^b)(\nabla_b Z^a)} - X^c Y^b \nabla_c \nabla_b Z^a \quad \Big| \text{ Leibniz}$$

$$+ \cancel{(Y^c \nabla_c X^b)(\nabla_b Z^a)} + Y^c X^b \nabla_c \nabla_b Z^a$$

$$= -X^b Y^c \underbrace{(\nabla_b \nabla_c - \nabla_c \nabla_b) Z^a}_{-R_{bcd}{}^a Z^d} = R_{bcd}{}^a X^b Y^c Z^d \quad \square$$

$\boxed{4}$ We know $\mathcal{L}_X$ satisfies the Lebnitz rule.

$$\mathcal{L}_X(ST) = (\mathcal{L}_X S)T + S(\mathcal{L}_X T).$$

$$\to D(ST) = (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X - \mathcal{L}_{[X,Y]})(ST)$$

$$= \mathcal{L}_X(\mathcal{L}_Y S)T + \mathcal{L}_X S\mathcal{L}_Y T - \mathcal{L}_Y(\mathcal{L}_X S)T - \mathcal{L}_Y S\mathcal{L}_X T - (\mathcal{L}_{[X,Y]}S)T - S(\mathcal{L}_{[X,Y]}T)$$

$$= (\mathcal{L}_X\mathcal{L}_Y S)T + \cancel{(\mathcal{L}_Y S)(\mathcal{L}_X T)} + \cancel{(\mathcal{L}_X S)(\mathcal{L}_Y T)} + S(\mathcal{L}_X\mathcal{L}_Y T) - (\mathcal{L}_Y\mathcal{L}_X S)T - \cancel{(\mathcal{L}_X S)(\mathcal{L}_Y T)}$$
$$\cancel{-(\mathcal{L}_Y\mathcal{L}_X S)T} - \cancel{(\mathcal{L}_Y S)(\mathcal{L}_X T)} - S(\mathcal{L}_Y\mathcal{L}_X T) - (\mathcal{L}_{[X,Y]}S)T - S(\mathcal{L}_{[X,Y]}T)$$

$$= (\mathcal{L}_X\mathcal{L}_Y S - \mathcal{L}_Y\mathcal{L}_X S - \mathcal{L}_{[X,Y]}S)T$$
$$+ S(\mathcal{L}_X\mathcal{L}_Y T - \mathcal{L}_Y\mathcal{L}_X T - \mathcal{L}_{[X,Y]}T)$$

$$= (DS)T + S(DT) \Rightarrow \text{Lebnitz property obeyed} \quad \square$$

$$\mathcal{L}_X f = X^a\partial_a f = X^a\nabla_a f$$

$$\to Df = \mathcal{L}_X\mathcal{L}_Y f - \mathcal{L}_Y\mathcal{L}_X f - \mathcal{L}_{[X,Y]}f$$

$$= \mathcal{L}_X Y^a\nabla_a f - \mathcal{L}_Y X^a\nabla_a f - [X,Y]^a\nabla_a f$$

$$= X^b\nabla_b Y^a\nabla_a f - Y^b\nabla_b X^a\nabla_a f$$
$$- (X^b\nabla_b Y^a)(\nabla_a f) + Y^b\nabla_b X^a(\nabla_a f)$$

$$= \cancel{(X^b\nabla_b Y^a)(\nabla_a f)} + \cancel{X^b Y^a\nabla_b\nabla_a f} + X^b Y^a\nabla_b\nabla_a f$$
$$- \cancel{(Y^b\nabla_b X^a)(\nabla_a f)} - Y^b X^a\nabla_b\nabla_a f$$
$$- \cancel{(X^b\nabla_b Y^a)(\nabla_a f)} + \cancel{(Y^b\nabla_b X^a)}\nabla_a f$$

$$= (\nabla_b \nabla_a f)(X^b Y^a - X^a Y^b)$$

$$= X^b Y^a \underbrace{(\nabla_b \nabla_a - \nabla_a \nabla_b)}_{=0} f = 0 \quad \square$$

$$\mathcal{L}_X Y^a = [X, Y]^a.$$

$$\therefore D Z^a = \mathcal{L}_X \mathcal{L}_Y Z^a - \mathcal{L}_Y \mathcal{L}_X Z^a - \mathcal{L}_{[X,Y]} Z^a.$$

$$= [X, [Y, Z]]^a - [Y, [X, Z]]^a - [[X, Y], Z]^a.$$

$$= [X, [Y, Z]]^a + [Y, [Z, X]]^a + [Z, [X, Y]]^a.$$

Jacobi's Identity. $\} = 0 \quad \square$

Consider tensor $T^{a_1 \cdots a_p}{}_{b_1 \cdots b_q}$, ~~vectors~~ $\cancel{A^{a_1}, A^{a_2}, \ldots A^{a_p}}$, ~~Covectors~~ $\cancel{B_a}$ B vector A^a, covector B_b

$$\underbrace{0}_{Df=0} = D(T^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} A^{b_1} \cdots A^{b_q} B_{a_1} \cdots B_{a_p}).$$

Lebnitz rule

$$= (D T^{a_1 \cdots a_p}{}_{b_1 \cdots b_q}) A^{b_1} \cdots A^{b_q} B_{a_1} \cdots B_{a_p}$$

$$+ T^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} (D A^{b_1}) A^{b_2} \cdots A^{b_q} B_{a_1} \cdots B_{a_p}$$

$$+ \cdots + T^{a_1 \cdots b_p}{}_{b_1 \cdots b_q} A^{b_1} \cdots A^{b_2} B_{a_1} \cdots B_{a_{p-1}} D B_{a_p}.$$

good

$$\because \quad D A^{b_1} = 0 \qquad D B_{a_i} = 0$$

$$\therefore \quad 0 = D(T^{a_1 \cdots a_p}{}_{b_1 \cdots b_q}) A^{b_1} \cdots A^{b_q} B_{a_1} \cdots B_{a_p} \quad \forall A, B.$$

$$\therefore \quad DT = 0 \text{ for any tensor } T \quad \forall X, Y.$$

$$\Rightarrow \quad D \equiv 0 \quad \forall X, Y \quad \square$$

$\boxed{5}$ Consider vector S ~~...~~ better to denote vectors with different letters

(1) $\mathcal{L}_X (R_{abcd} S^a S^b S^c S^d) = \cancel{X^e \partial_e}\; X^e \nabla_e (R_{abcd} S^a S^b S^c S^d)$

$\mathcal{L}_X(\text{scalar}) = X^e \nabla_e(\text{scalar})$

$= X^e(\nabla_e R_{abcd}) \cdot S^a S^b S^c S^d + \cancel{R_{abcd} X^e \nabla_e}\; R_{abcd} S^b S^c S^d X^e \nabla_e S^a$

$+ R_{abcd} S^a S^c S^d X^e \nabla_e S^b + R_{abcd} S^a S^b S^d X^e \nabla_e S^c$

$+ R_{abcd} S^a S^b S^c X^e \nabla_e S^d$

On the other hand, by Lebnitz's rule:

(2) $\mathcal{L}_X (R_{abcd} S^a S^b S^c S^d)$

$= S^a S^b S^c S^d \mathcal{L}_X R_{abcd} + R_{abcd} S^b S^c S^d \mathcal{L}_X S^a$

$+ R_{abcd} S^a S^c S^d \mathcal{L}_X S^b + R_{abcd} S^a S^b S^d \mathcal{L}_X S^c$

$+ R_{abcd} S^a S^b S^c \mathcal{L}_X S^d$

$= S^a S^b S^c S^d \mathcal{L}_X R_{abcd} + R_{abcd} S^b S^c S^d X^e \nabla_e S^a$

$- R_{abcd} S^e S^b S^c S^d \nabla_e X^a + R_{abcd} S^a S^c S^d X^e \nabla_e S^b$

$- R_{abcd} S^e S^a S^c S^d \nabla_e X^b$ + similar terms ...

equating (1) = (2) :

terms like $R_{abcd} S^b S^c S^d X^e \nabla_e S^a$ cancel

~~$S^a S^b S^c$~~ and terms like

$R_{abcd} S^e S^b S^c S^d \nabla_e X^a = \underbrace{R_{ebcd} S^a S^b S^c S^d \nabla_a X^e}_{\text{swap } e \leftrightarrow a}$

So we have

$S^aS^bS^cS^d \mathcal{L}_X R_{abcd}$

$= S^aS^bS^cS^d (X^e \nabla_e R_{abcd} + R_{ebcd} \nabla_a X^e + R_{aecd} \nabla_b X^e$
$+ R_{abed} \nabla_c X^e + R_{abce} \nabla_d X^e)$

$\forall\ S$ good

This proves the required result. $\square$

By definition of Riemann curvature tensor we have

① $[\nabla_a, \nabla_b] \nabla_c X_d = R_{abcd} \nabla^e X_d + R_{abde} \nabla_c X^e \quad \forall\ X$

For X a Killing vector, use $\nabla_a \nabla_b X_c = R_{cbad} X^d$ gives

② $[\nabla_a, \nabla_b] \nabla_c X_d = \nabla_a \nabla_b \nabla_c X_d - \nabla_b \nabla_a \nabla_c X_d$
$= \nabla_a (R_{dcbe} X^e) - \nabla_b (R_{dcae} X^e)$
$= X^e \nabla_a R_{dcbe} + R_{dcbe} \nabla_a X^e - X^e \nabla_b R_{dcae}$
$- R_{dcae} \nabla_b X^e$

①, ② are equal $\Rightarrow$

$\rightarrow R_{abce} \nabla^e X_d = - R_{abce} \nabla_d X^e \quad \because \nabla_{[a} X_{b]} = 0 \ \forall$ Killing X

$\rightarrow X^e (\nabla_a R_{dcbe} - \nabla_b R_{dcae}) \underset{\text{symmetry}}{=} X^e (\nabla_a R_{bedc} - \nabla_b R_{aedc}).$
$\underset{\text{antisymmetry}}{=} - X^e (\nabla_a R_{ebdc} + \nabla_b R_{aedc}) \underset{\text{first Bianchi}}{=} + X^e \nabla_e R_{badc}$

$= X^e \nabla_e R_{abcd}$

so : $X^e \nabla_e R_{abcd} + R_{dcbe} \nabla_a X^e - R_{dcae} \nabla_b X^e$
$= - R_{abce} \nabla_d X^e + R_{abde} \nabla_c X^e$

$\Rightarrow$ $X^e \nabla_e R_{abcd} + R_{ebcd} \nabla_a X^e + R_{aecd} \nabla_b X^e$
$+ R_{abed} \nabla_c X^e + R_{abce} \nabla_d X^e = 0$

But the above is precisely $\mathcal{L}_X R_{abcd}$ so
$\mathcal{L}_X R_{abcd} = 0$ if X is a Killing vector $\square$

$\boxed{6}$ If we want to know the value of a vector field X in any point in spacetime, we should speify ~~the value~~ at a particular point the values of X, ~~first~~ ∇X, $\nabla\nabla X$, $\nabla\nabla\nabla X$, ... (X and all deritives of X up to arbitrary order).

For a Killing vector field, however, because $\nabla_a \nabla_b X_c = - R_{bcad} X^d$, if we know X ~~we~~ at a particular point we ~~obta~~ automatically know $\nabla\nabla X$, the second derivatives of X. And if we know ∇X, then $\because$ $\nabla\nabla\nabla X \sim \nabla R X \sim \nabla X$ so we know 3rd derivatives. etc. As long as we speify X and ∇X at a point, we know X in all points of spacetime. good

So to count number of linearly independent X^a, we need to count how many X and ∇X components we can set freely.

$\rightarrow$ For X_a, $\because$ space is n-dimensional so we have n choices.

$\rightarrow$ For $\nabla_a X_b$, $\because$ $\nabla_{(a} X_{b)} = 0$ $\therefore$ $\nabla_a X_a = 0$ (No sum) we need distinct $a \neq b$ and once we know $\nabla_a X_b$ we know $\nabla_b X_a = -\nabla_a X_b$. This gives ~~a~~

$$\binom{n}{2} = \frac{n(n-1)}{2}$$ choices. good

$\rightarrow$ In total, $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ choices $\square$

$\mathcal{L}_X R_{abcd} = 0$ from $\boxed{Q5}$

$\Rightarrow X^e \nabla_e R_{abcd} + R_{ebcd} \nabla_a X^e + R_{aecd} \nabla_b X^e + R_{abed} \nabla_c X^e$
$+ R_{abce} \nabla_d X^e = 0$

$\Rightarrow (\nabla_e R_{abcd}) X^e + (R_{ebcd} \delta^f{}_a + R_{aecd} \delta^f{}_b + R_{abed} \delta^f{}_c$
$+ R_{abce} \delta^f{}_d) \nabla_f X^e = 0$ $(*)$

In most general case second bracket should be symmetric (not necessarily 0)

But after taking the trace the exact same result is produced.

see weinberg pp 380-383

In a space that has maximum possible of Killing vectors, all components of X and ∇X can be choosen ~~fre~~ freely to be non-zero. (In fact technically $\nabla_f X^e = 0$ when $f = e$ ~~but~~; in that case X^e can still be choose to be non-zero and $(A) X + (B)(0) = 0$ doesn't imply $B = 0$. But we ignore this and do the calculation anyway and later argue the result is the only correct result possible).

$\Rightarrow$ Quantities in the brackets ( ) are 0

$\Rightarrow \nabla_e R_{abcd} = 0$ $\square$ $\Rightarrow$ curvature is same everywhere.

what does this say about ∇R, R, etc?

$\Rightarrow$ try $R_{ebcd} \delta^f{}_a + R_{aecd} \delta^f{}_b + R_{abed} \delta^f{}_c + R_{abce} \delta^f{}_d = 0$

take trace $\underbrace{R_{ebcd} \delta^f{}_a \delta^a{}_f}_{= \delta^f{}_f = n} + R_{aecd} \delta^f{}_b \delta^a{}_f + R_{abed} \delta^f{}_c \delta^a{}_f$
$+ R_{abce} \delta^f{}_d \delta^a{}_f = 0$

$\Rightarrow n R_{ebcd} + R_{becd} + R_{cbed} + R_{dbce} = 0$

$\rightarrow n R_{ebcd} + \underbrace{(R_{becd} + R_{cbed} + R_{ecbd})}_{0} = 0$

$\Rightarrow R_{ebcd} = 0$ This of course is a solution, but obviously not the most general one. We get this because our assumption that ( ) in front of $\nabla_e X^e$ is 0 is wrong. We need to rewrite (*) so that this assumption is right.

$\because \delta = \hat{g}g$
$\therefore \nabla\delta = 0$

Using $\nabla_{[a}X_{b]} = 0$ we rewrite $\mathcal{L}_X R_{abcd} = 0$ ~~as~~ with $\nabla_e R_{abcd} = 0$ as

$$R_{ebcd}\nabla_a X^e + R_{aecd}\nabla_b X^e - R_{abfd}\nabla^f X_c - R_{abcf}\nabla^f X_d = 0$$

$$\Rightarrow 0 = (\underbrace{R_{ebcd}}_{=R_{dcbe}}\delta^f{}_a + \underbrace{R_{aecd}}_{=-R_{eacd}=-R_{dcae}}\delta^f{}_b)\nabla_f X^e - (\underbrace{R_{abfd}}_{=-R_{abdf}}\delta^e_c + R_{abcf}\delta^e_d)\nabla^f X_e$$

$\cancel{\Rightarrow 0 = (R_{dcbe}}$

$$\Rightarrow 0 = (R_{dcb}{}^e\delta^f{}_a - R_{dca}{}^e\delta^f{}_b + R_{abd}{}^f\delta^e_c - R_{abc}{}^f\delta^e_d)\nabla_f X_e$$

and try
$$R_{dcb}{}^e\overbrace{\delta^f{}_a\delta^a_f}^{n} - R_{dca}{}^e\overbrace{\delta^f{}_b\delta^a_f}^{\delta^a_b} + R_{abd}{}^f\delta^e_c\delta^a_f - R_{abc}{}^f\delta^e_d\delta^a_f = 0$$

$$\Rightarrow nR_{dcb}{}^e - R_{dcb}{}^e - R_{bad}{}^a\delta^e_c + R_{bac}{}^a\delta^e_d = 0$$

$$\Rightarrow (n-1)R_{dcb}{}^e - R_{bd}\delta^e_c + R_{bc}\delta^e_d = 0$$

$$\times g_{ae} \Rightarrow (n-1)\underbrace{R_{dcba}}_{=R_{cdab}=R_{abcd}} - g_{ac}R_{bd} + g_{ad}R_{bc} = 0 \quad (**)$$

~~contracting c,a~~

$\cancel{\Rightarrow (n-1)R_{db}}$

contracting d, b ($\times g^{db}$):

$$(n-1)R_{ca} - g_{ac}R + \underbrace{R_{ac}}_{=R_{ca}} = 0 \Rightarrow \underline{R_{ac} = \frac{1}{n}g_{ac}R} \quad (***)$$

put (***) back to (**) gives

$$(n-1)R_{abcd} - \frac{1}{n}R\, g_{ac} g_{bd} + \frac{1}{n}R\, g_{ad} g_{bc} = 0$$

$$\Rightarrow \quad R_{abcd} = \frac{R}{n(n-1)}\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right) \quad (\star)$$

this solution is the most general solution, because $\nabla_e R_{abcd} = 0 \Rightarrow R_{abcd}$ is a combination of the metric tensor, and $g_{ac}g_{bd} - g_{ad}g_{bc}$ is a unique combination of g that gives the same symmetry properties of $R_{abcd} \Rightarrow R_{abcd} \propto g_{ac}g_{bd} - g_{ad}g_{bc}$

proportionality constant $\frac{R}{n(n-1)}$ is ~~it~~ indeed a constant since $\nabla_e R_{abcd} = 0 \Rightarrow R$ is constant.

In this case our assumption that $(\quad)\nabla_f X_e \Rightarrow (\quad)$ vanishes is in fact correct because using this assumption we get the most general ~~solution and~~ form of solution, and this form of solution makes $(\quad)^{ef}_{abcd}$ vanishes automatically when $e=f$ without having to ~~have~~ solve an equation when $e=f$.

$\Rightarrow$ R_{abcd} is given by $(\star)$ $\square$

In 4-D minkowski metric $R_{abcd} = 0$, $R = 0$

$\Rightarrow$ $\nabla_a \nabla_b X_c = -R_{bcad} X^d$ gives $\nabla_a \nabla_b X_c = 0$

$\forall$ Killing vector X_c, also $\nabla_a X_b + \nabla_b X_a = 0$

$\forall$ Killing vector X_a.

(L.I. = linearly independent)

In $n=4$ dimensions there are at most $\frac{4(5)}{2} = 10$ L.I. Killing vectors, ~~so~~ and $\because$ $\nabla_a \nabla_b X_c = 0$, $\nabla_a X_b + \nabla_b X_a = 0$.

$\therefore$ we can find 4 L.I. Killing vectors already - The constant vectors.

$\Rightarrow$ ~~P1~~ $P_1 = (1,0,0,0)$, $P_2 = (0,1,0,0)$, $P_3 = (0,0,1,0)$, $P_4 = (0,0,0,1)$

Also in Minkowski space affine connection (Levi-Civita) vanishes $\therefore$ $\nabla = \partial$ $\Rightarrow$ $\begin{cases} \partial_a \partial_b X_c = 0 \\ \partial_a X_b + \partial_b X_a = 0 \end{cases}$

$\partial_a \partial_b X_c = 0$ means X doesn't have ~~quadratic~~ quadratic or higher power terms, so beside the 4 constant Killing vectors, we have some other Killing vectors linear in x^a, and obeys $\partial_a X_b + \partial_b X_a = 0$

$\because X$ is linear $\therefore$ $\partial_a X_b$ is a number, so we choose $\partial_a X_b = 1$, $\partial_b X_a = -1$ and find another 6 L.I. Killing vectors (and remember $\partial_a X_a = 0$ (no sum))

$M_{12} = (0, y, -x, 0)$ $\qquad$ $M_{23} = (0, 0, z, -y)$

$M_{31} = (0, -z, 0, x)$ $\qquad$ ~~$M_{01} = (-t, x, 0, 0)$~~

~~$M_{02} = (-t, 0, y, 0)$~~ $\qquad$ ~~$M_{03} = (-t, 0, 0, z)$~~

$M_{01} = (x, -t, 0, 0)$

$M_{02} = (y, 0, -t, 0)$ $\qquad$ $M_{03} = (z, 0, 0, -t)$

translations	rotations	"Boosts"
P_1, P_2, P_3, P_4	M_{12}, M_{23}, M_{31}	M_{01}, M_{02}, M_{03}

State $I(\mathbb{I})$ here where $\mathbb{I} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\boxed{7}$ $\mathbb{R}^3$:

$I = z\partial_y - y\partial_z = (0, z, -y)$. $J = x\partial_z - z\partial_x$
$= (-z, 0, x)$

$K = y\partial_x - x\partial_y = (y, -x, 0)$

Conserved quantity associated with I is

$I_a \dot{x}^a = zV_y - yV_z = \dfrac{(\vec{P} \times \vec{X})_x}{m}$ = angular momentum

per unit mass in x-direction, similarly for

J and K

$\Rightarrow$ I, J, K represent rotations I in x-direction,
J in y and K in z. $\square$

$I(f) = I_a \partial^a f$ metric $g^{ab} = g_{ab} = \delta^a{}_b = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$\Rightarrow [I, J](f) = I(J(f)) - J(I(f))$

$= (z\partial_y - y\partial_z)(x\partial_z f - z\partial_x f) - (x\partial_z - z\partial_x)(z\partial_y f - y\partial_z f)$
$= zx\cancel{\partial_y\partial_z f} - \cancel{z^2\partial_y\partial_x f} - y\cancel{x\partial_z\partial_z f} + \cancel{yz\partial_z\partial_x f} + y\partial_x f$
$- \cancel{xz\partial_z\partial_y f} + \cancel{xy\partial_z\partial_z f}$
$+ \cancel{z^2\partial_y\partial_x f} - \cancel{zy\partial_x\partial_z f} - x\partial_y f$

$= (y\partial_x - x\partial_y)f = K(f) \Rightarrow [I, J] = K$ $\square$

similarly $[J, K]f = (x\partial_z - z\partial_x)(y\partial_x - x\partial_y)f - (y\partial_x - x\partial_y)(x\partial_z - z\partial_x)f$
$= (z\partial_y - y\partial_z)f = If \Rightarrow [J, K] = I$

and $[K, I](f) = (y\partial_x - x\partial_y)(z\partial_y - y\partial_z)f - (z\partial_y - y\partial_z)(y\partial_x - x\partial_y)f$
$= (x\partial_z - z\partial_x)f = J(f)$
$\Rightarrow [K, I] = J$